

Classical inverse scattering [mln104]

Goal: reconstruction of potential $V(r)$ from cross section $\sigma(\theta)$.

Assumptions: $dV/dr < 0$ (repulsive force), $V(0) > E$, $V(\infty) = 0$.

Consequence: $\theta(s_1) > \theta(s_2)$ if $s_1 < s_2$.

Calculate $s(\theta)$ from $\sigma(\theta)$: $2\pi \int_{\theta}^{\pi} d\theta' \sin \theta' \sigma(\theta') = 2\pi \int_0^s ds' s' = \pi s^2$.

Orbital integral from [mln20] with $u \doteq 1/r$:

$$\frac{\pi - \theta(s)}{2} = \int_0^{u_m} \frac{sdu}{\sqrt{1 - \frac{V(1/u)}{E} - s^2u^2}}, \quad s^2u_m^2 + V(1/u_m)/E = 1.$$

Transformation with $x \doteq 1/s^2$, $\tilde{\theta}(x) = \theta(s)$, and $w(u) \doteq \sqrt{1 - V(1/u)/E}$:

$$\frac{\pi - \tilde{\theta}(x)}{2} = \int_0^{u_m} \frac{du}{\sqrt{x[w(u)]^2 - u^2}}, \quad u_m^2 = x[w(u_m)]^2 \Rightarrow u_m(x).$$

Transformation: $\frac{1}{2} \int_0^{\alpha} dx \frac{\pi - \tilde{\theta}(x)}{\sqrt{\alpha - x}} = \int_0^{\alpha} dx \int_0^{u_m} \frac{du}{\sqrt{(x[w(u)]^2 - u^2)(\alpha - x)}}$.

$$\Rightarrow \pi\sqrt{\alpha} - \int_0^{\alpha} dx \tilde{\theta}'(x)\sqrt{\alpha - x} = \pi \int_0^{u_m(\alpha)} \frac{du}{w(u)}. \quad [\text{mex243}]$$

Set $\alpha = u^2/w^2$, implying $u_m(\alpha) \rightarrow u$, take d/du , and multiply by du :

$$\frac{\pi}{w} dw = -d\left(\frac{u}{w}\right) \int_0^{u^2/w^2} dx \frac{\tilde{\theta}'(x)}{\sqrt{u^2/w^2 - x}}. \quad [\text{mex244}]$$

Integrate differentials dw and $d(u/w)$ with consistent boundary values:

$$\Rightarrow w(u) = \exp\left(\frac{1}{\pi} \int_{w/u}^{\infty} ds \frac{\theta(s)}{\sqrt{s^2 - [w(u)]^2/u^2}}\right). \quad [\text{mex245}]$$

The solution $w(u)$ of this integral equation for given $\theta(s)$ thus determines $V(r)$ from $\sigma(\theta)$.

[Landau and Lifshitz 1976]