Noether's Theorem II [mln13]

A more general class of symmetry transformations leaves the Lagrange equations invariant but not the Lagrangian itself.

Consider again a Lagrangian system $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

Theorem (more general case):

If a transformation $Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon),\ i = 1, \ldots, n$ with

$Q_i = q_i$ at $\epsilon = 0$ can be found such that

$$\left.\frac{\partial L'}{\partial \epsilon}\right|_{\epsilon=0} = \frac{d}{dt}\left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0}$$

is satisfied, where

$$L'(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon) \doteq L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$

and $F(Q_1, \ldots, Q_n, t, \epsilon)$ is an arbitrary differentiable function, then the following quantity is conserved:

$$I(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \left.\frac{\partial q_i}{\partial \epsilon}\right|_{\epsilon=0} - \left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0}.$$

Proof:

Use inverse transformation $q_i(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon),\ i = 1, \ldots, n$ and keep the variables $Q_i, \dot{Q}_i$ fixed. Then use gauge invariance (see [mex21]),

$$L'(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon) = L(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t) + \frac{d}{dt} F(Q_1, \ldots, Q_n, t, \epsilon)$$

and the steps of the proof in [mln12].

$$\left[\frac{\partial L'}{\partial \epsilon} - \frac{d}{dt}\frac{\partial F}{\partial \epsilon}\right]_{\epsilon=0} = \frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \epsilon} - \frac{\partial F}{\partial \epsilon}\right]_{\epsilon=0}.$$

Applications:

- pure Galilei transformation [mex37]