

Limit Cycles [mln74]

Not all attractors in 2D phase flow are fixed points.
There exists exactly one other type of attractor: the limit cycle.

Example: Flow in (x, y) -plane with circular limit cycle.

Equations of motion in polar coordinates (r, θ) :

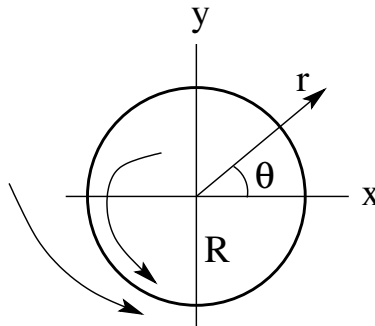
$$\dot{r} = -\alpha r(r - R), \quad \dot{\theta} = \omega = \text{const.}$$

Periodic trajectory: $r(t) = R = \text{const.}$, $\theta(t) = \theta_0 + \omega t$.

Linearized radial equation of motion for $|r(t) - R| \doteq s \ll R$:

$$\dot{s} = -\alpha R s \quad \Rightarrow \quad s(t) = s_0 e^{-\alpha R t}.$$

\Rightarrow Periodic trajectory is an attractor (limit cycle).



Note presence of fixed point at $r = 0$:

Cartesian coordinates: $x = r \cos \theta$, $y = r \sin \theta$.

Linear analysis of equations of motion predict spiral repellor:

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta = -\alpha (\sqrt{x^2 + y^2} - R) x - \omega y,$$

$$\dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta = -\alpha (\sqrt{x^2 + y^2} - R) y + \omega x,$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} \alpha R & -\omega \\ +\omega & \alpha R \end{pmatrix} \quad \Rightarrow \quad \lambda = \alpha R \pm i\omega.$$