Lagrange equations derived from D'Alembert's principle [mln8]

N particles, k holonomic constraints, $3N-k$ degrees of freedom.

Coordinates: $\mathbf{r}_i(q_1,\ldots,q_{3N-k},t) \quad : \; i=1,\ldots,N.$

Velocities: $\dot{\mathbf{r}}_i(q_1,\ldots,q_{3N-k},\dot{q}_1,\ldots,\dot{q}_{3N-k},t) \quad : \; i=1,\ldots,N.$

Forces: $\mathbf{F}_i(\mathbf{r}_1,\ldots,\mathbf{r}_N,\dot{\mathbf{r}}_1,\ldots,\dot{\mathbf{r}}_N,t) \quad : \; i=1,\ldots,N.$

D'Alembert's equation: $\displaystyle\sum_{j=1}^{3N-k}\left[\sum_{i=1}^{N}(m_i\ddot{\mathbf{r}}_i-\mathbf{F}_i)\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right]\delta q_j=0$

Transformations of two terms:

- $\displaystyle\sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=Q_j \quad : \; j=1,\ldots,3N-k.$

  $\displaystyle Q_j(q_1,\ldots,q_{3N-k},\dot{q}_1,\ldots,\dot{q}_{3N-k},t)=\sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}.$

- Use $\displaystyle\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)=m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}+m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial qj}$

  $\displaystyle\Rightarrow\quad\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial q_j}\right].$

  Use $\displaystyle\frac{d\mathbf{r}_i}{dt}=\sum_{l=1}^{3N-k}\frac{\partial\mathbf{r}_i}{\partial q_l}\dot{q}_l+\frac{\partial\mathbf{r}_i}{\partial t}=\dot{\mathbf{r}}_i(q_j,\dot{q}_j,t)\;\Rightarrow\;\frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j}.$

  $\displaystyle\Rightarrow\quad\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j}\right)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial q_j}\right].$

  Use $\displaystyle\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_j}\left(\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)\right]=\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j}\right).$

  Use $\displaystyle\frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)=m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial q_j}.$

  $\displaystyle\Rightarrow\quad\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)\right]-\frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right).$

Kinetic energy: $\displaystyle T(q_1,\ldots,q_{3N-k},\dot{q}_1,\ldots,q_{3N-k},t)=\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2.$

D'Alembert's equation for generalized coordinates:

$$\Rightarrow \sum_{j=1}^{3N-k} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0$$

Virtual displacements δq_j are independent. The content of each parenthesis must vanish.

Lagrange equations in general form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad : \; j = 1, \ldots, 3N - k.$$

If all forces can be derived from a scalar potential, we can construct a Lagrangian as follows:

Assumption: $\mathbf{F}_i = -\nabla_i \tilde{V}, \quad \tilde{V}(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = V(q_1, \ldots, q_{3N-k}, t).$

$$\Rightarrow \; Q_j = Q_j(q_1, \ldots, q_{3N-k}, t) = -\frac{\partial V}{\partial q_j}, \quad \frac{\partial V}{\partial \dot{q}_j} = 0.$$

Lagrangian: $L \doteq T(q_1, \ldots, q_{3N-k}, \dot{q}_1, \ldots, q_{3N-k}, t) - V(q_1, \ldots, q_{3N-k}, t).$

The Lagrange equations can then be stated more compactly as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad : \; j = 1, \ldots, 3N - k.$$

Generalized momenta: $p_j \doteq \dfrac{\partial L}{\partial \dot{q}_j} \quad : \; j = 1, \ldots, 3N - k.$