Point transformations [mln88]

Point transformations $q_i \to Q_i$ are invertible coordinate transformations in configuration space.

Given a set of transformation relations between generalized coordinates:

$$q_i = q_i(Q_1, \ldots, Q_n; t), \quad i = 1, \ldots, n.$$

Generalized velocities: $\dot{q}_i(Q_1, \ldots, Q_n; \dot{Q}_1, \ldots, \dot{Q}_n; t) = \sum_j \frac{\partial q_j}{\partial Q_j} \dot{Q}_j + \frac{\partial q_j}{\partial t}$.

Transformed Lagrangian via substitution of transformation relations:

$$\tilde{L}(Q_1, \ldots, Q_n; \dot{Q}_1, \ldots, \dot{Q}_n; t) = L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t).$$

Invariance of Lagrange equations under point transformations [mex79]:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \Leftrightarrow \quad \frac{\partial \tilde{L}}{\partial Q_i} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}_i} = 0.$$

Canonical momenta: $p_i \doteq \frac{\partial L}{\partial \dot{q}_i}, \quad P_i \doteq \frac{\partial \tilde{L}}{\partial \dot{Q}_i}$.

Relation between canonical momenta [mex80]: $P_i = \sum_j p_j \frac{\partial q_j}{\partial Q_i}$.

Relation between Hamiltonians [mex80]:

$$\tilde{H}(Q_1, \ldots, Q_n, P_1, \ldots, P_n, t) = H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) - \sum_j p_j \frac{\partial q_j}{\partial t}.$$

Invariance of canonical equations under point transformations [mex82]:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Leftrightarrow \quad \dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i}.$$