Hamilton's Principal Function [mln96]

We seek a canonical transformation $H(q,p,t) \;\rightarrow\; K(Q,P) \equiv 0$.
Here q stands for $q_1,\ldots,q_n$ etc.

Canonical equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \rightarrow \quad \dot{Q}_i = \frac{\partial K}{\partial P_i} = 0, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0.$$

Hamilton's principal function: $S(q,P,t)$.

- S is the F_2-type generating function of this canonical transformation.
- S depends on $n+1$ variables $q_1,\ldots,q_n,t$ and n parameters $P_1,\ldots,P_n$.
- $p_j = \dfrac{\partial S}{\partial q_j}, \quad Q_j = \dfrac{\partial S}{\partial P_j}, \quad K - H = \dfrac{\partial S}{\partial t}$.

Hamilton-Jacobi equation: $H\left(q_1,\ldots,q_n;\dfrac{\partial S}{\partial q_n},\ldots,\dfrac{\partial S}{\partial q_n};t\right) + \dfrac{\partial S}{\partial t} = 0$.

- First-order partial differential equation for $S(q,P,t)$.
- Integration constants $P_1,\ldots,P_n$ plus additive constant.
- From the solution $S(q,P,t)$ we infer the transformation relations:

$$p_j(q,P,t) = \frac{\partial S}{\partial q_j}, \quad Q_j(p,P,t) = \frac{\partial S}{\partial P_j} = \text{const.}$$

$$\Rightarrow \; q_j(Q,P,t), \quad p_j(Q,P,t).$$

- The constants Q_j, P_j are functions of the initial values $q_j^{(0)}, p_j^{(0)}$.

Physical significance of Hamilton's principal function:

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j}\dot{q}_j + \frac{\partial S}{\partial t} = \sum_j p_j\dot{q}_j - H = L.$$

For fixed initial conditions Hamilton's principal function becomes the action.