

Hamilton's Characteristic Function [mln97]

Two distinct ways of solving the Hamilton-Jacobi equation become available when the Hamiltonian does not explicitly depend on time.

If $H(q, p) = E = \text{const.}$ then $\frac{\partial S}{\partial t} = -E = \text{const.}$

Set $S(q, P, t) = W(q, P) - Et$.

Hamilton's characteristic function: $W(q_1, \dots, q_n; P_1, \dots, P_n)$.

Method #1:

- Employ the ansatz, $S(q, P, t) = W(q, P) - Et$, for Hamilton's principal function.
- The Hamilton-Jacobi equation to be solved thus reduces as follows:

$$H\left(q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0 \quad \Rightarrow \quad H\left(q, \frac{\partial W}{\partial q}\right) - E = 0.$$

- Proceed as in [mln96] with $S(q, P, t) = W(q, P) - Et$.
- One of the integration constants is reserved: $P_1 = E$.

Method #2:

- Solve the Hamilton-Jacobi equation $H\left(q, \frac{\partial W}{\partial q}\right) - E = 0$.
- $W(q, P)$ is a F_2 -type generating function of a canonical transformation to action-angle coordinates with $P_1 = K(P) = E$.
- Canonical Equations: $\dot{Q}_j = \frac{\partial K}{\partial P_j} = \delta_{j1}$, $\dot{P}_j = -\frac{\partial K}{\partial Q_j} = 0$.
- Solution: $P_j = \text{const.}$, $Q_j = t\delta_{j1} + Q_j^{(0)}$.
- Transformation to original canonical coordinates:

$$Q_j = \frac{\partial}{\partial P_j} W(q, P), \quad p_j = \frac{\partial}{\partial q_j} W(q, P).$$
$$\Rightarrow q_j = q_j(Q^{(0)}, P, t), \quad p_j = p_j(Q^{(0)}, P, t).$$