Toda System (integrable) [msl12]

Particle of unit mass moving in (x_1, x_2)-plane.

First integral of the motion: $E(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + V(x_1, x_2)$,

$$V(x_1, x_2) = \frac{1}{24}\left[\exp\left(2x_2 + 2\sqrt{3}x_1\right) + \exp\left(2x_2 - 2\sqrt{3}x_1\right) + \exp\left(-4x_2\right)\right] - \frac{1}{8}.$$

Second integral of the motion:

$$\begin{aligned} K(x_1, x_2, \dot{x}_1, \dot{x}_2) &= 8\dot{x}_1\left(\dot{x}_1^2 - 3\dot{x}_2^2\right) + \left(\dot{x}_1 + \sqrt{3}\dot{x}_2\right)\exp\left(2x_2 - 2\sqrt{3}x_1\right) \\ &- 2\dot{x}_1\exp\left(-4x_2\right) + \left(\dot{x}_1 - \sqrt{3}\dot{x}_2\right)\exp\left(2x_2 + 2\sqrt{3}x_1\right). \end{aligned}$$

Equations of motion: $\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2, \quad \dot{y}_1 = -\frac{\partial V}{\partial x_1}, \quad \dot{y}_2 = -\frac{\partial V}{\partial x_2}.$

Phase space of integrable system is densely foliated by both rational and irrational tori. Irrational (rational) tori have measure one (zero).

A randomly chosen initial condition lies with probability one on an irrational torus and, therefore, with probability zero on a rational torus.

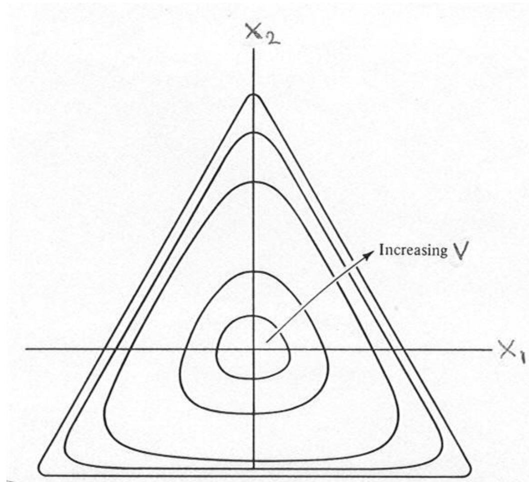
Irrational tori contain one quasiperiodic trajectory. Rational tori contain infinitely many periodic trajectories.

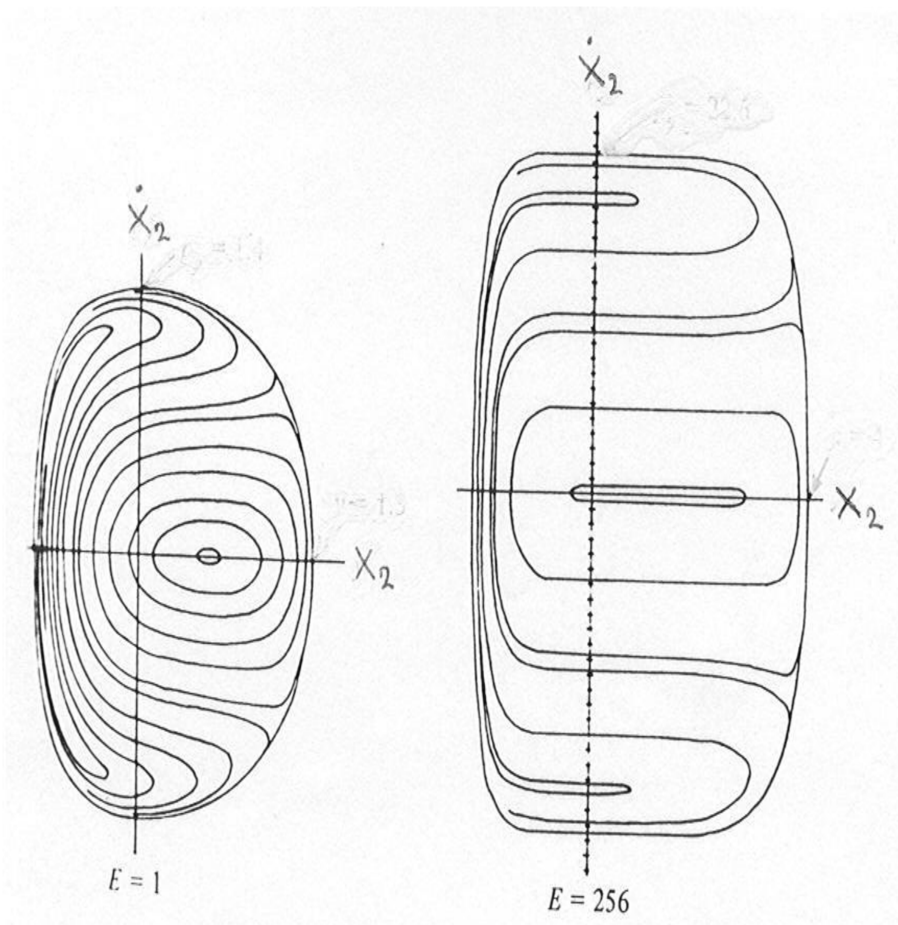
Each torus, rational or irrational, is fully specified by the values of the two actions J_1, J_2.

Poincaré surface of section: for a given phase-space trajectory, plot only points with $x_1 = 0$ and $\dot{x}_1 > 0$. Project those points onto the $(x_2, \dot{x}_2)$-plane.

On the Poincaré surface of section, tori are nonintersecting closed lines. Quasiperiodic trajectories produce infinitely many points that gradually fill in the line that represents an irrational torus. Periodic trajectories fill in a finite number of points on the line that represents a rational torus.

The structure of lines on a Poincaré cut pertaining to an integrable system evokes the Hamiltonian phase flow in the phase plane of a system with one degree of freedom. What looks like elliptic or hyperbolic fixed point are, in fact, special tori reduced to periodic trajectories.





[Lichtenberg and Lieberman 1983]