Hénon-Heiles System (nonintegrable) [msl13]

Particle of unit mass moving in (x_1, x_2)-plane.

First integral of the motion: $E(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + V(x_1, x_2)$,

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + x_1^2 x_2 - \frac{1}{3}x_2^3$$

The Hénon-Heiles potential is the Toda potential expanded about $(0, 0)$ up to cubic terms.

A second integral of the motion $K(x_1, x_2, \dot{x}_1, \dot{x}_2)$ does not exist.

Equations of motion: $\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2, \quad \dot{y}_1 = -\frac{\partial V}{\partial x_1}, \quad \dot{y}_2 = -\frac{\partial V}{\partial x_2}.$

Each trajectory is confined to a 3D energy manifold $E(x_1, x_2, \dot{x}_1, \dot{x}_2) = \text{const}$ in 4D phase space.

The Cauchy nonintersection condition of trajectories is a much milder constraint on a 3D manifold than on a 2D torus. This opens the door for trajectories that are neither periodic nor quasiperiodic.

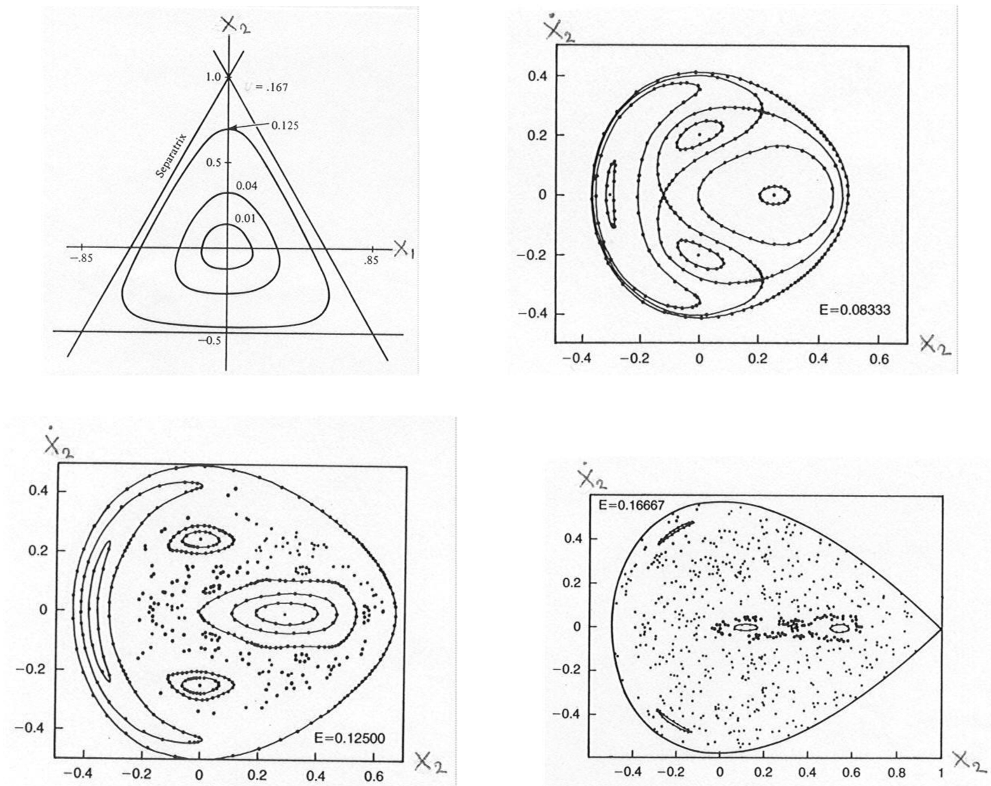
Tori still exist, but only irrational tori. They are no longer dense anywhere in phase space. However, there is a dense set of periodic trajectories. They are the remnants of the rational tori, which do no longer exist.

The regions between intact (irrational) tori is populated by chaotic trajectories in addition to periodic trajectories. Chaotic trajectories explore 3D regions on any given energy manifold.

Neither the intact tori nor the chaotic trajectories are dense, but they share the full measure. The periodic trajectories, which are dense throughout phase space, have zero measure.

Chaotic trajectories are highly sensitive to initial conditions. Trajectories with infinitesimally different initial conditions exhibit an exponential divergence from each other.

In some regions of phase space (e.g. near elliptic fixed points on the Poincaré cut), chaos is severely confined between intact tori. In other regions (e.g. near hyperbolic fixed points) chaos is more widespread.



[Lichtenberg and Lieberman 1983]

Chaos undermines the concept of determinism that we commonly attribute to classical mechanics.

The exponential sensitivity on initial conditions of the canonical equations in the case of chaotic solutions has serious implications. If it takes an initial condition of n digits precision to compute a chaotic trajectory over a certain time interval to preset accuracy, then it takes an initial condition of $2n$ digits precision to calculate the same trajectory over twice the time interval to the same accuracy.

In other words, the trajectory itself as a data set of digitized coordinates is not longer than the data set that contains the initial conditions necessary to compute it. From this perspective, the equations of motion look utterly redundant. Their predictive power has disappeared.