Eulerian Angular Velocities [msl26]

The rotation of a rigid body is described by the vector $\vec{\omega}$ of angular velocity. In general, this vector changes magnitude and direction in both coordinate systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3).

The most natural formulation of the equations of motion for a rigid body is in the body frame (x_1, x_2, x_3). They are called Euler's equations.

However, the solution is incomplete unless we know how to express the vector $\vec{\omega}$ in the frame (x'_1, x'_2, x'_3), which is typically the frame of the observer.

Eulerian angular velocities:

$\dot{\phi}$ directed along x'_3-axis.
$\dot{\theta}$ directed along line of nodes.
$\dot{\psi}$ directed along x_3-axis.

Projections onto axes of (x_1, x_2, x_3):

$$\dot{\psi}_1 = 0,\ \dot{\psi}_2 = 0,\ \dot{\psi}_3 = \dot{\psi}.$$
$$\dot{\theta}_1 = \dot{\theta}\cos\psi,\ \dot{\theta}_2 = -\dot{\theta}\sin\psi,\ \dot{\theta}_3 = 0.$$
$$\dot{\phi}_1 = \dot{\phi}\sin\theta\sin\psi,\ \dot{\phi}_2 = \dot{\phi}\sin\theta\cos\psi,\ \dot{\phi}_3 = \dot{\phi}\cos\theta.$$

Projections onto axes of (x'_1, x'_2, x'_3):

$$\dot{\phi}'_1 = 0,\ \dot{\phi}'_2 = 0,\ \dot{\phi}'_3 = \dot{\phi}.$$
$$\dot{\theta}'_1 = \dot{\theta}\cos\phi,\ \dot{\theta}'_2 = \dot{\theta}\sin\phi,\ \dot{\theta}'_3 = 0.$$
$$\dot{\psi}'_1 = \dot{\psi}\sin\theta\sin\phi,\ \dot{\psi}'_2 = -\dot{\psi}\sin\theta\cos\phi,\ \dot{\psi}'_3 = \dot{\psi}\cos\theta.$$

Instantaneous angular velocity in the frame (x_1, x_2, x_3): $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$.

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi.$$
$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi.$$
$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi}.$$

Instantaneous angular velocity in the frame (x'_1, x'_2, x'_3): $\vec{\omega}' = (\omega'_1, \omega'_2, \omega'_3)$.

$$\omega'_1 = \dot{\phi}'_1 + \dot{\theta}'_1 + \dot{\psi}'_1 = \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi.$$
$$\omega'_2 = \dot{\phi}'_2 + \dot{\theta}'_2 + \dot{\psi}'_2 = -\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi.$$
$$\omega'_3 = \dot{\phi}'_3 + \dot{\theta}'_3 + \dot{\psi}'_3 = \dot{\psi}\cos\theta + \dot{\phi}.$$

Magnitude of angular velocity: $|\vec{\omega}|^2 = |\vec{\omega}'|^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta.$