Torque-Free Motion of Symmetric Top [msl27]

Inertial coordinate system: (x_1', x_2', x_3') with origin at the center of mass.
Body coordinate system: (x_1, x_2, x_3) with principal axes.
Principal moments of inertia: $I_1 = I_2 \equiv I_\perp \neq I_3$.
The symmetry axis x_3 is called figure axis. Its direction is $\hat{\mathbf{e}}_3$.

Calculation performed in body frame of reference.

Euler's equations:
$$\begin{aligned} I_\perp \dot{\omega}_1 - (I_\perp - I_3)\omega_2\omega_3 &= 0 \\ I_\perp \dot{\omega}_2 - (I_3 - I_\perp)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

Invariant: $\dot{\omega}_3 = 0 \quad \Rightarrow \; \omega_3 = \text{const.}$

Linear ODEs: $\dot{\omega}_1 + \Omega\omega_2 = 0, \quad \dot{\omega}_2 - \Omega\omega_1 = 0; \qquad \Omega \equiv \dfrac{I_3 - I_\perp}{I_\perp}\omega_3$

Solution: $\omega_1(t) = \omega_\perp \cos(\Omega t), \quad \omega_2(t) = \omega_\perp \sin(\Omega t).$

Angular velocity: $\vec{\omega} = (\omega_1, \omega_2, \omega_3), \quad \omega_\perp = \sqrt{\omega_1^2 + \omega_2^2}, \quad \omega = \sqrt{\omega_\perp^2 + \omega_3^2}.$

Angular momentum: $L_1 = I_\perp\omega_1(t), \quad L_2 = I_\perp\omega_2(t), \quad L_3 = I_3\omega_3 = \text{const.}$

Vectors $\mathbf{L}$ and $\vec{\omega}$ precess uniformly and in phase about figure axis x_3.

Motion can be described as $\mathbf{L}$-cone rolling about $\hat{\mathbf{e}}_3$-cone:
- clockwise on the outside if $I_3 < I_\perp$,
- counterclockwise on the inside if $I_3 > I_\perp$.

View from inertial frame of reference.

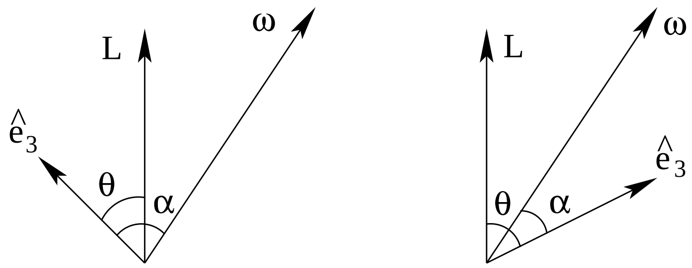
Zero torque implies $\mathbf{L}$ =const.

Vectors $\vec{\omega}$ and $\hat{\mathbf{e}}_3$ precess uniformly and in phase about vector $\mathbf{L}$.

Motion can be described as $\hat{\mathbf{e}}_3$-cone rolling about $\mathbf{L}$-cone:
- counterclockwise on the outside if $I_3 < I_\perp$,
- counterclockwise on the inside if $I_3 > I_\perp$.

$$\tan\alpha = \frac{\omega_\perp}{\omega_3}.$$

$$\tan\theta = \frac{L_\perp}{L_3} = \frac{I_\perp \omega_\perp}{I_3 \omega_3} = \frac{I_\perp}{I_3}\,\tan\alpha.$$



$I_3 > I_\perp$ $\qquad$ $I_3 < I_\perp$

$\alpha > \theta$ $\qquad$ $\alpha < \theta$