

## Torque-Free Motion of Asymmetric Top [msl28]

Inertial coordinate system:  $(x'_1, x'_2, x'_3)$  with origin at the center of mass.

Body coordinate system:  $(x_1, x_2, x_3)$  with principal axes.

Principal moments of inertia:  $I_1 < I_2 < I_3$ .

Conservation laws:

- Angular momentum:  $L^2 = L_1^2 + L_2^2 + L_3^2 = \text{const}$  (sphere)
- Energy:  $E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} = \text{const}$  (ellipsoid)

The figure shows one ellipsoid for some value of  $E$ . Also shown are lines of intersection of the ellipsoid with spheres corresponding to different values of  $L^2$ .

View of motion from body frame of reference:

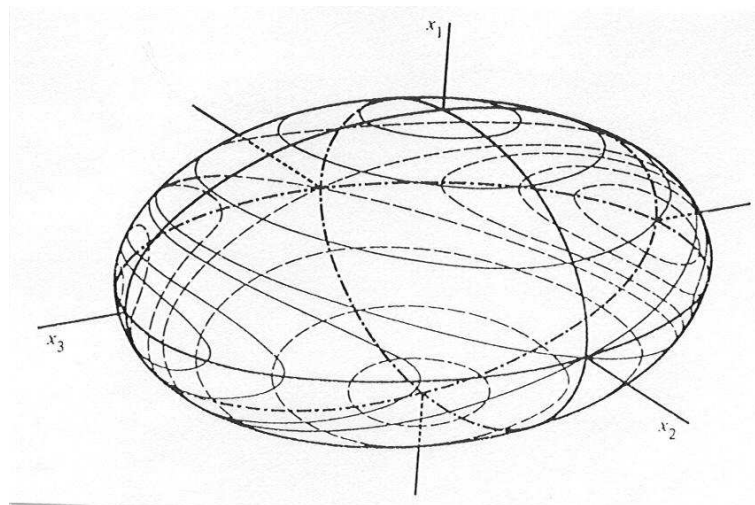
Ellipsoid is at rest and fixed to the rigid body.

Tip of vector  $\mathbf{L}$  traces one line of intersection on ellipsoid.

View of motion from inertial frame of reference.

Vector  $\mathbf{L}$  is fixed in space.

Rigid body rotates such that the tip of  $\mathbf{L}$  traces one line of intersection on ellipsoid.



Motion in time for  $I_1 < I_2 < I_3$

Euler's equations:  $I_i \dot{\omega}_i = \omega_j \omega_k (I_j - I_k)$ ,  $\{i, j, k\} = \text{cycl}\{1, 2, 3\}$ .

Inverse moments of inertia:  $J_i \equiv 1/I_i$ ,  $i = 1, 2, 3$ .

Euler's equations for  $L_i = I_i \omega_i$ :  $\dot{L}_i = L_j L_k (J_k - J_j)$ .

Solution:  $L_1(t) = a_1 \text{dn}(\Omega t, k)$ ,  $L_2(t) = a_2 \text{sn}(\Omega t, k)$ ,  $L_3(t) = a_3 \text{cn}(\Omega t, k)$ .

Amplitudes:  $a_1^2 = \frac{2E - J_3 L^2}{J_1 - J_3}$ ,  $a_2^2 = \frac{J_1 L^2 - 2E}{J_1 - J_2}$ ,  $a_3^2 = \frac{J_1 L^2 - 2E}{J_1 - J_3}$ .

Initial conditions are specified by invariants  $E$  and  $L$  and a phase angle.

Condition:  $J_3 L^2 < 2E < J_1 L^2$ .

Modulus:  $k^2 = \frac{J_2 - J_3}{J_1 - J_2} \frac{J_1 L^2 - 2E}{2E - J_3 L^2}$ .

Period of the motion:  $T = 4K(k)/\Omega$ ,  $\Omega^2 = (J_1 - J_2)(2E - J_3 L^2)$ .

$K(k)$ : complete elliptic integral of the first kind.

Jacobian elliptic functions:

$$\begin{aligned} \frac{d}{dt} \text{sn}(t, k) &= \text{cn}(t, k) \text{dn}(t, k) \\ \frac{d}{dt} \text{cn}(t, k) &= -\text{sn}(t, k) \text{dn}(t, k) \\ \frac{d}{dt} \text{dn}(t, k) &= -k^2 \text{sn}(t, k) \text{cn}(t, k) \end{aligned}$$

For initial conditions which lead to  $k^2 > 1$ , use the relations  
 $\text{cn}(t, k) = \text{dn}(kt, k^{-1})$ ,  $\text{sn}(t, k) = k^{-1} \text{sn}(kt, k^{-1})$ ,  $\text{dn}(t, k) = \text{cn}(kt, k^{-1})$

Restoring symmetry ( $k \rightarrow 0$ ):  $\text{dn}(t, k) \rightarrow 1$ ,  $\text{sn}(t, k) \rightarrow \sin t$ ,  $\text{cn}(t, k) \rightarrow \cos t$ .

