

Poisson Brackets [msl30]

Definition: $\{f, g\} = \sum_{j=1}^n \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j} \frac{\partial f}{\partial p_j} \right),$

where $f(q_1, \dots, q_n, p_1, \dots, p_n)$, $g(q_1, \dots, q_n, p_1, \dots, p_n)$ are arbitrary dynamical variables expressed as functions of canonical coordinates.

Algebraic properties:

- $\{f, g\} = -\{g, f\},$
- $\{f, c\} = 0$ if $c = \text{const.},$
- $\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\},$
- $\{f_1 f_2, g\} = f_1 \{f_2, g\} + f_2 \{f_1, g\},$
- $\frac{\partial}{\partial t} \{f, g\} = \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\},$
- $\{q_j, f\} = \frac{\partial f}{\partial p_j}, \quad \{p_j, f\} = -\frac{\partial f}{\partial q_j}.$

Fundamental Poisson brackets: $\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$

Invariance under canonical transformations:

$$Q_j = Q_j(q_1, \dots, q_n, p_1, \dots, p_n), \quad P_j = P_j(q_1, \dots, q_n, p_1, \dots, p_n)$$
$$\Rightarrow \{Q_i, Q_j\}_{q,p} = 0, \quad \{P_i, P_j\}_{q,p} = 0, \quad \{Q_i, P_j\}_{q,p} = \delta_{ij}$$

Canonical equations: $\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}.$

Jacobi's identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Poisson's theorem: $\frac{d}{dt} \{f, g\} = \left\{ \frac{df}{dt}, g \right\} + \left\{ f, \frac{dg}{dt} \right\}$ [mex191]

Integral of the motion: $\frac{df}{dt} = \{f, H\} = 0.$

If the functions f and g are integrals of the motion, then the function $\{f, g\}$ is also an integral of the motion.