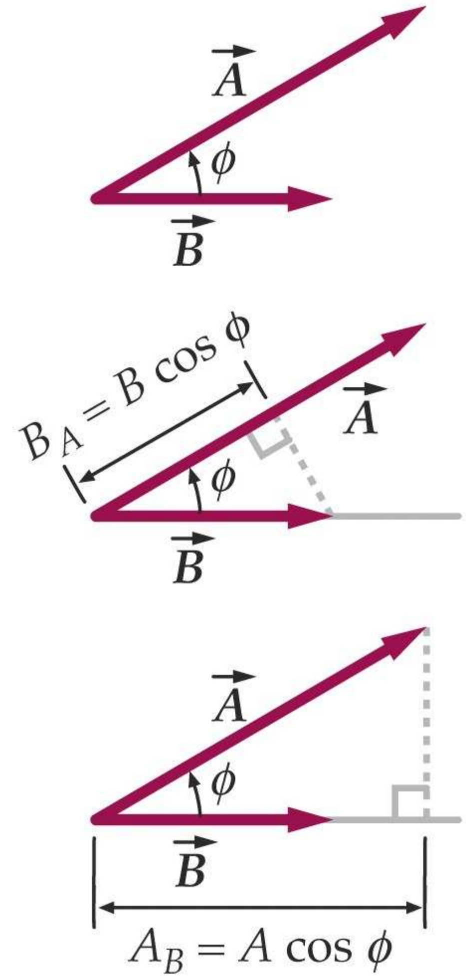
Dot Product Between Vectors

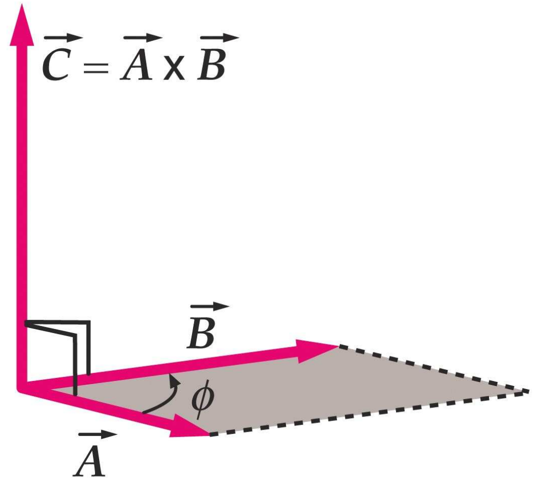
Consider two vectors $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$.

- $\vec{A}\cdot\vec{B} = AB\cos\phi = AB_A = BA_B$.
- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.
- $\vec{A}\cdot\vec{B} = AB$ if $\vec{A} \parallel \vec{B}$.
- $\vec{A}\cdot\vec{B} = 0$ if $\vec{A} \perp \vec{B}$.
- $$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y(\hat{i}\cdot\hat{j}) + A_xB_z(\hat{i}\cdot\hat{k}) \\ &\quad + A_yB_x(\hat{j}\cdot\hat{i}) + A_yB_y(\hat{j}\cdot\hat{j}) + A_yB_z(\hat{j}\cdot\hat{k}) \\ &\quad + A_zB_x(\hat{k}\cdot\hat{i}) + A_zB_y(\hat{k}\cdot\hat{j}) + A_zB_z(\hat{k}\cdot\hat{k}).\end{aligned}$$
- Use $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$,
  $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.
- $\Rightarrow \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$.



Cross Product Between Vectors

Consider two vectors $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$.

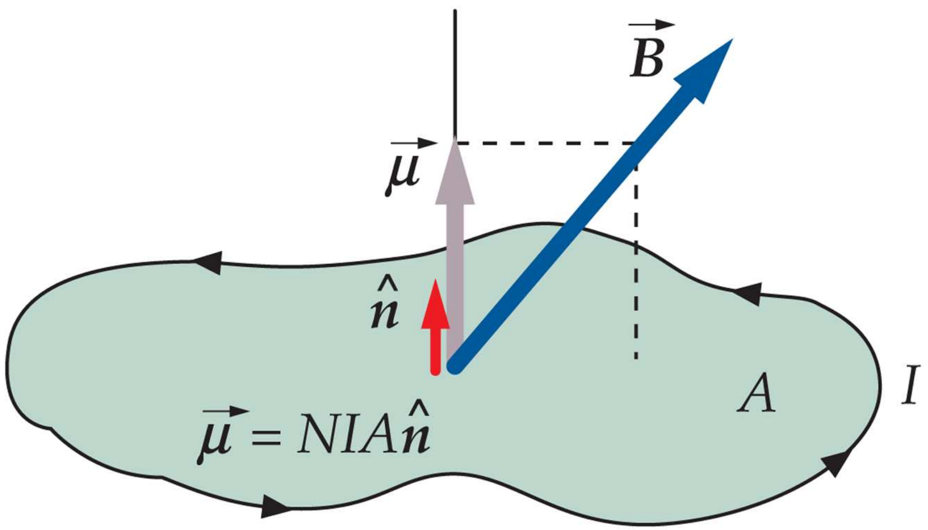


- $\vec{A} \times \vec{B} = AB\sin\phi\,\hat{n}$.
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.
- $\vec{A} \times \vec{A} = 0$.
- $\vec{A} \times \vec{B} = AB\,\hat{n}$ if $\vec{A} \perp \vec{B}$.
- $\vec{A} \times \vec{B} = 0$ if $\vec{A} \parallel \vec{B}$.
- $$\begin{aligned}\vec{A} \times \vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x(\hat{i}\times\hat{i}) + A_xB_y(\hat{i}\times\hat{j}) + A_xB_z(\hat{i}\times\hat{k}) \\ &\quad + A_yB_x(\hat{j}\times\hat{i}) + A_yB_y(\hat{j}\times\hat{j}) + A_yB_z(\hat{j}\times\hat{k}) \\ &\quad + A_zB_x(\hat{k}\times\hat{i}) + A_zB_y(\hat{k}\times\hat{j}) + A_zB_z(\hat{k}\times\hat{k}).\end{aligned}$$
- Use $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$,
  $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$.
- $\Rightarrow\ \vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$.

Magnetic Dipole Moment of Current Loop

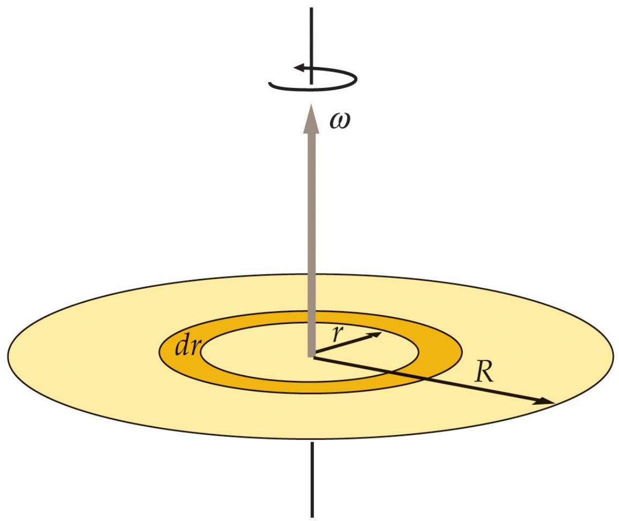
N: number of turns
I: current through wire
A: area of loop
$\hat{n}$: unit vector perpendicular to plane of loop
$\vec{\mu} = NIA\hat{n}$: magnetic dipole moment

$\vec{B}$: magnetic field
$\vec{\tau} = \vec{\mu} \times \vec{B}$: torque acting on current loop



Magnetic Moment of a Rotating Disk

Consider a nonconducting disk of radius R with a uniform surface charge density σ. The disk rotates with angular velocity $\vec{\omega}$.

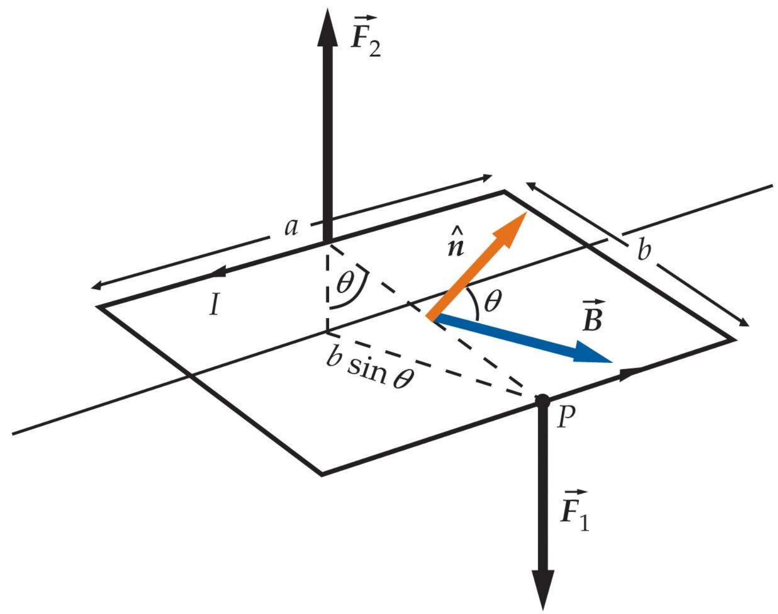


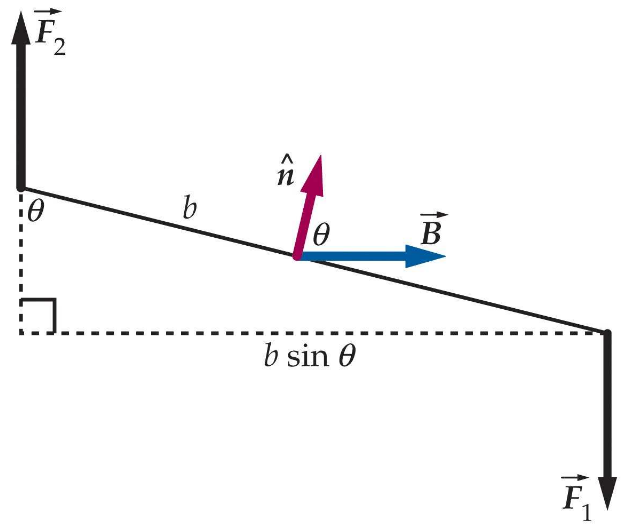
Calculation of the magnetic moment $\vec{\mu}$:

- Total charge on disk: $Q = \sigma(\pi R^2)$.
- Divide the disk into concentric rings of width dr.
- Period of rotation: $T = \dfrac{2\pi}{\omega}$.
- Current within ring: $dI = \dfrac{dQ}{T} = \sigma(2\pi r dr)\dfrac{\omega}{2\pi} = \sigma\omega r dr$.
- Magnetic moment of ring: $d\mu = dI(\pi r^2) = \pi\sigma\omega r^3 dr$.
- Magnetic moment of disk: $\mu = \displaystyle\int_0^R \pi\sigma\omega r^3 dr = \frac{\pi}{4}\sigma R^4\omega$.
- Vector relation: $\vec{\mu} = \dfrac{\pi}{4}\sigma R^4\vec{\omega} = \dfrac{1}{4}QR^2\vec{\omega}$.

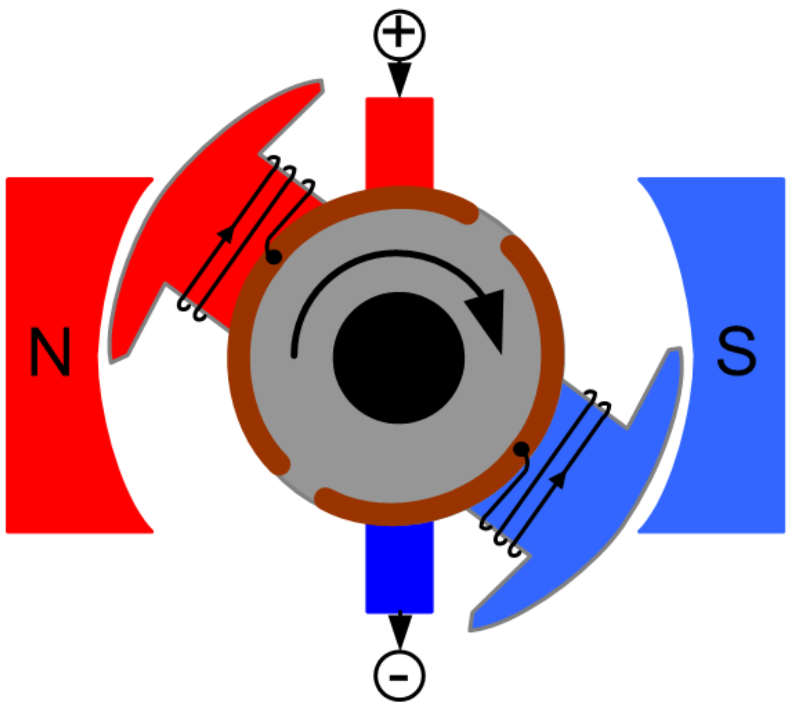
Torque on Current Loop

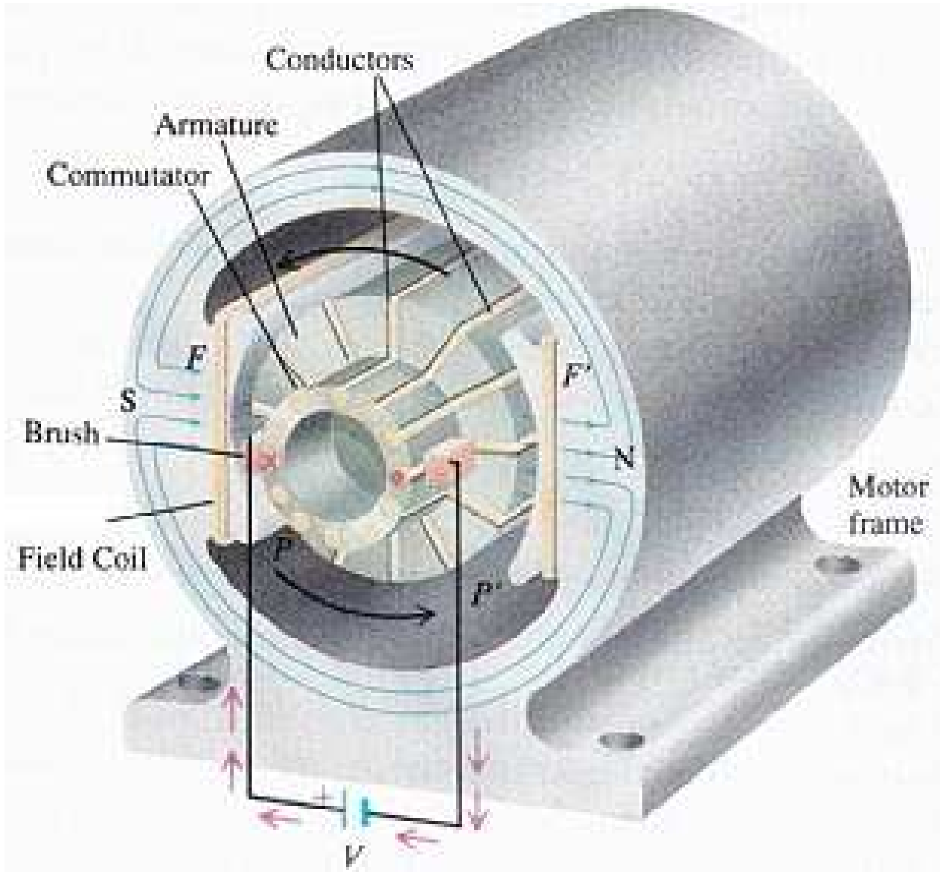
- magnetic field: $\vec{B}$ (horizontal)
- area of loop: $A = ab$
- unit vector $\perp$ to plane of loop: $\hat{n}$
- right-hand rule: $\hat{n}$ points up.
- forces on sides a: $F = IaB$ (vertical)
- forces on sides b: $F = IbB$ (horizontal, not shown)
- torque: $\tau = Fb\sin\theta = IAB\sin\theta$
- magnetic moment: $\vec{\mu} = IA\hat{n}$
- torque (vector): $\vec{\tau} = \vec{\mu} \times \vec{B}$





Direct-Current Motor

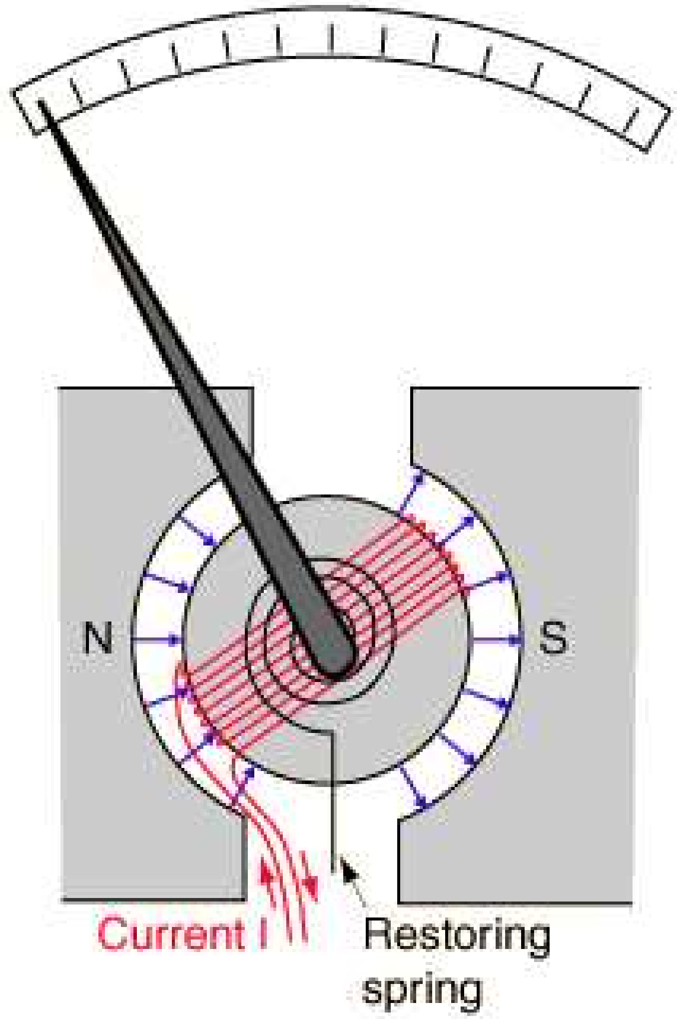




Galvanometer

Measuring direct currents.

- magnetic moment $\vec{\mu}$ (along needle)
- magnetic field $\vec{B}$ (toward right)
- torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ (into plane)

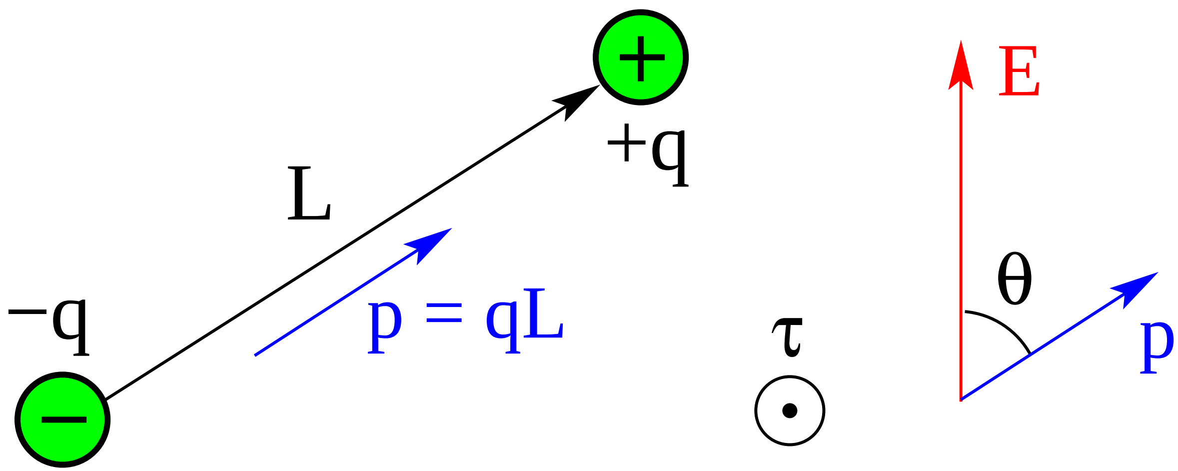


Electric Dipole in Uniform Electric Field

- Electric dipole moment: $\vec{p} = q\vec{L}$
- Torque exerted by electric field: $\vec{\tau} = \vec{p} \times \vec{E}$
- Potential energy: $U = -\vec{p} \cdot \vec{E}$

$$U(\theta) = -\int_{\pi/2}^{\theta} \tau(\theta) d\theta = pE \int_{\pi/2}^{\theta} \sin\theta d\theta = -pE\cos\theta$$

Note: $\tau(\theta)$ and $d\theta$ have opposite sign.

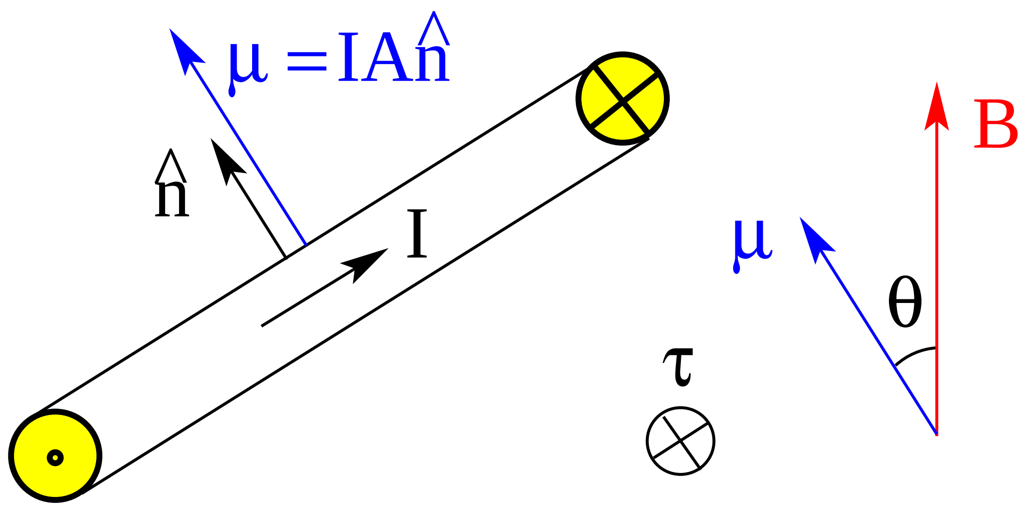


Magnetic Dipole in Uniform Magnetic Field

- Magnetic dipole moment: $\vec{\mu} = IA\hat{n}$
- Torque exerted by magnetic field: $\vec{\tau} = \vec{\mu} \times \vec{B}$
- Potential energy: $U = -\vec{\mu} \cdot \vec{B}$

$$U(\theta) = -\int_{\pi/2}^{\theta} \tau(\theta) d\theta = \mu B \int_{\pi/2}^{\theta} \sin\theta d\theta = -\mu B \cos\theta$$

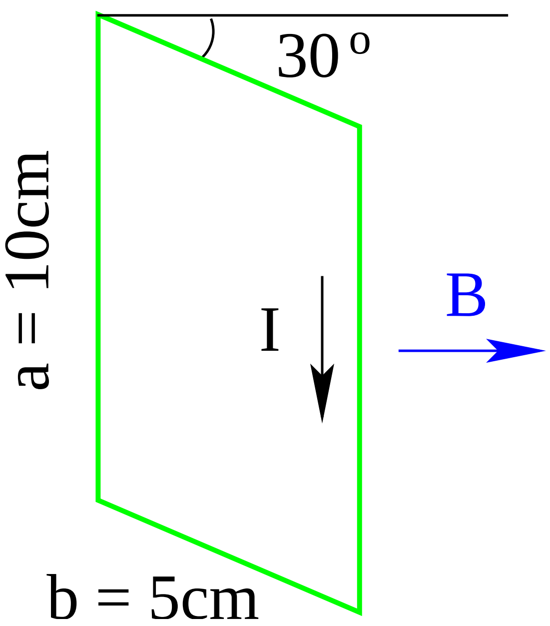
Note: $\tau(\theta)$ and $d\theta$ have opposite sign.



Magnetic Force Application (7)

The rectangular 20-turn loop of wire is 10cm high and 5cm wide.
It carries a current $I = 0.1$A and is hinged along one long side.
It is mounted with its plane at an angle of 30° to the direction of a uniform magnetic field of magnitude $B = 0.50$T.

- Calculate the magnetic moment μ of the loop.
- Calculate the torque τ acting on the loop about the hinge line.



Magnetic Force Application (4)

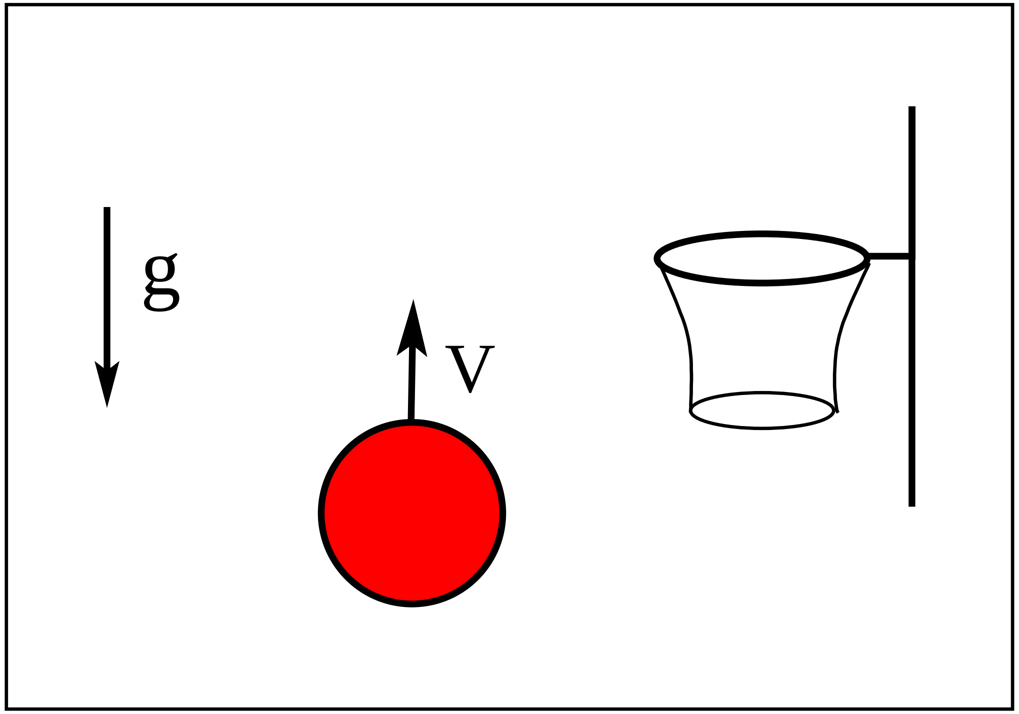
A negatively charged basketball is thrown vertically up against the gravitational field $\vec{g}$.

Which direction of

(a) a uniform electric field $\vec{E}$,

(b) a uniform magnetic field $\vec{B}$

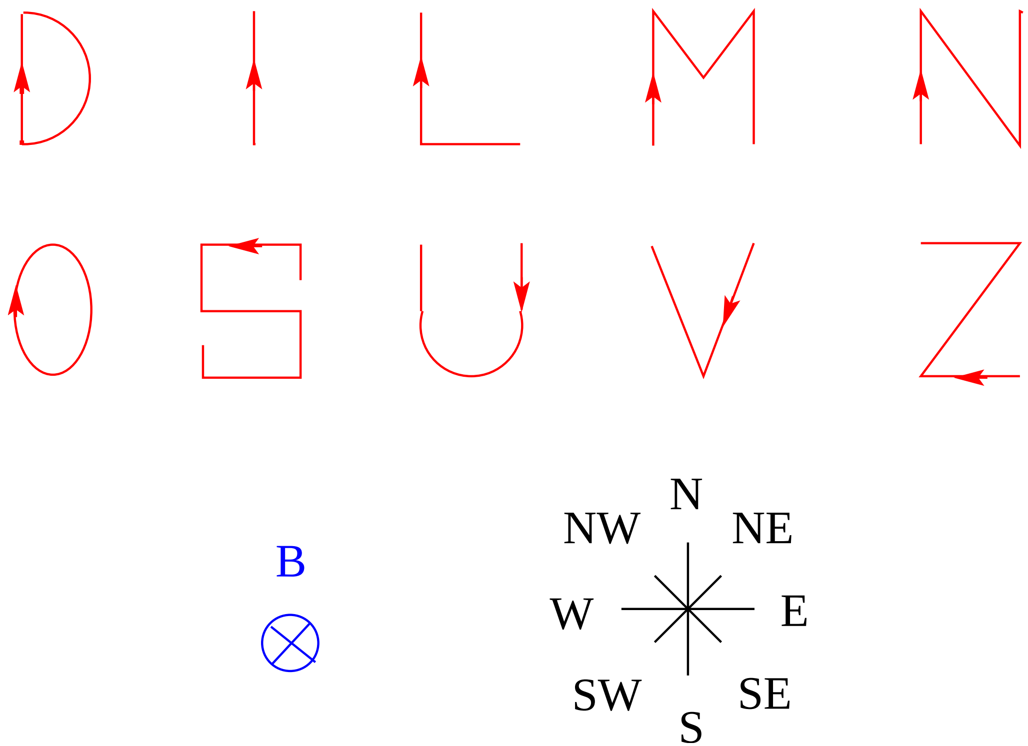
will give the ball a chance
to find its way into the basket?
(up/down/left/right/back/front)



Magnetic Force Application (6)

An electric current flows through each of the letter-shaped wires in a region of uniform magnetic field pointing into the plane.

- Find the direction of the resultant magnetic force on each letter.



A triangular current loop is free to rotate around the vertical axis PQ.

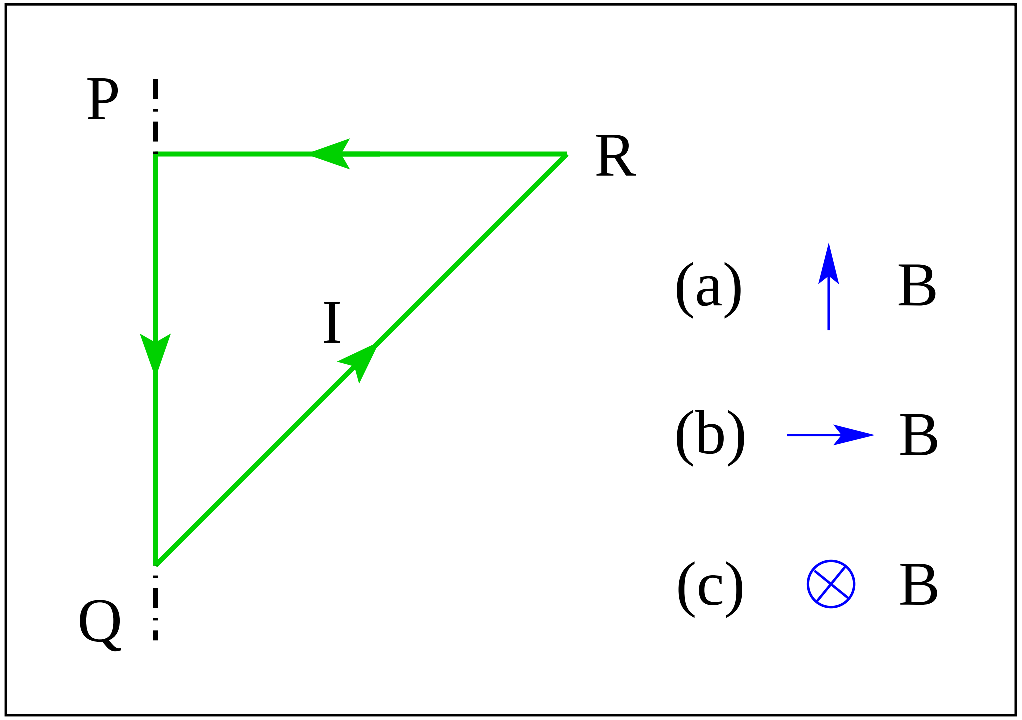
If a uniform magnetic field $\vec{B}$ is switched on, will the corner R of the triangle start to move out of the plane, into the plane, or will it not move at all?

Find the answer for a field $\vec{B}$ pointing

(a) up,

(b) to the right,

(c) into the plane.

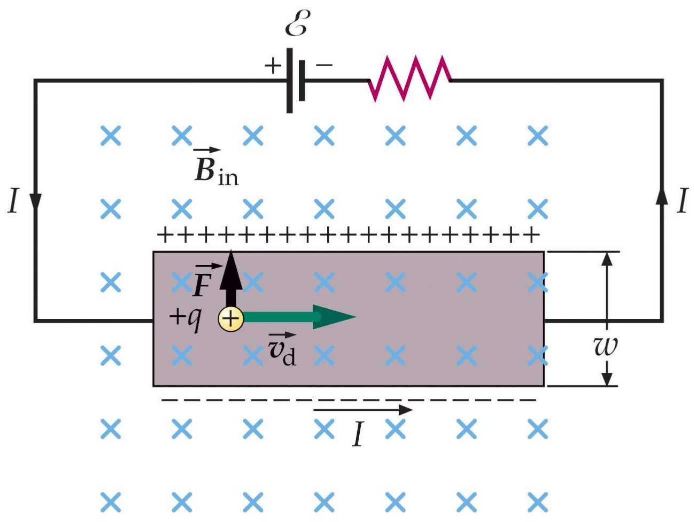


Hall Effect

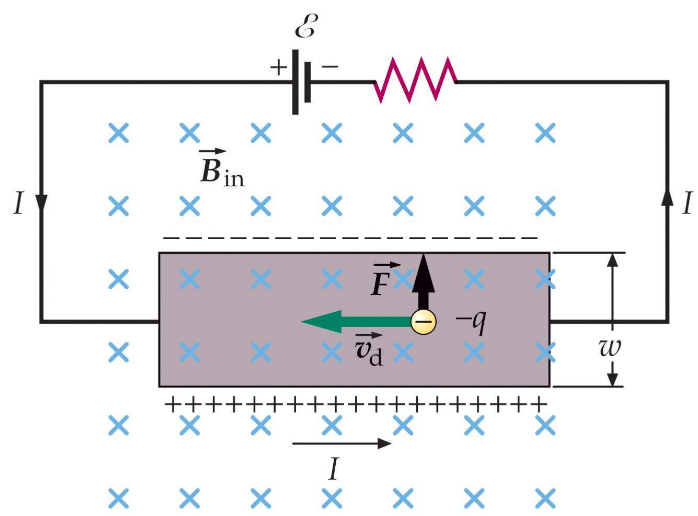
Method for dermining whether charge carriers are positively or negatively charged.

- Magnetic field $\vec{B}$ pulls charge carriers to one side of conducting strip.
- Accumulation of charge carriers on that side and depletion on opposite side produce transverse electric field $\vec{E}$.
- Transverse forces on charge carrier: $F_E = qE$ and $F_B = qv_dB$.
- In steady state forces are balanced: $\vec{F}_E = -\vec{F}_B$.
- Hall voltage in steady state: $V_H = Ew = v_dBw$.

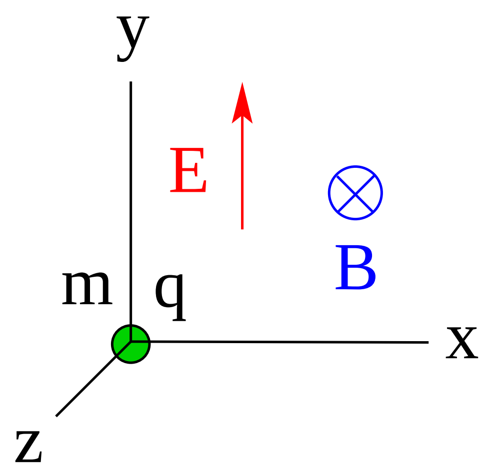
positive charge carriers



negative charge carriers



Charged Particle in Crossed Electric and Magnetic Fields (1)



- Release particle from rest.
- Force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
- (1) $F_x = m\dfrac{dv_x}{dt} = -qv_yB \quad \Rightarrow \quad \dfrac{dv_x}{dt} = -\dfrac{qB}{m}v_y$
- (2) $F_y = m\dfrac{dv_y}{dt} = qv_xB + qE \Rightarrow \dfrac{dv_y}{dt} = \dfrac{qB}{m}v_x + \dfrac{qE}{m}$
- Ansatz: $v_x(t) = w_x\cos(\omega_0 t) + u_x,\; v_y(t) = w_y\sin(\omega_0 t) + u_y$
- Substitute ansatz into (1) and (2) to find $w_x, w_y, u_x, u_y, \omega_0$.
- (1) $-\omega_0 w_x \sin(\omega_0 t) = -\dfrac{qB}{m}w_y\sin(\omega_0 t) - \dfrac{qB}{m}u_y$
- (2) $\omega_0 w_y \cos(\omega_0 t) = \dfrac{qB}{m}w_x\cos(\omega_0 t) + \dfrac{qB}{m}u_x + \dfrac{qE}{m}$
- $\Rightarrow\; u_y = 0, \quad u_x = -\dfrac{E}{B}, \quad \omega_0 = \dfrac{qB}{m}, \quad w_x = w_y \equiv w$
- Initial condition: $v_x(0) = v_y(0) = 0 \Rightarrow\; w = \dfrac{E}{B}$

Charged Particle in Crossed Electric and Magnetic Fields (2)

- Solution for velocity of particle:

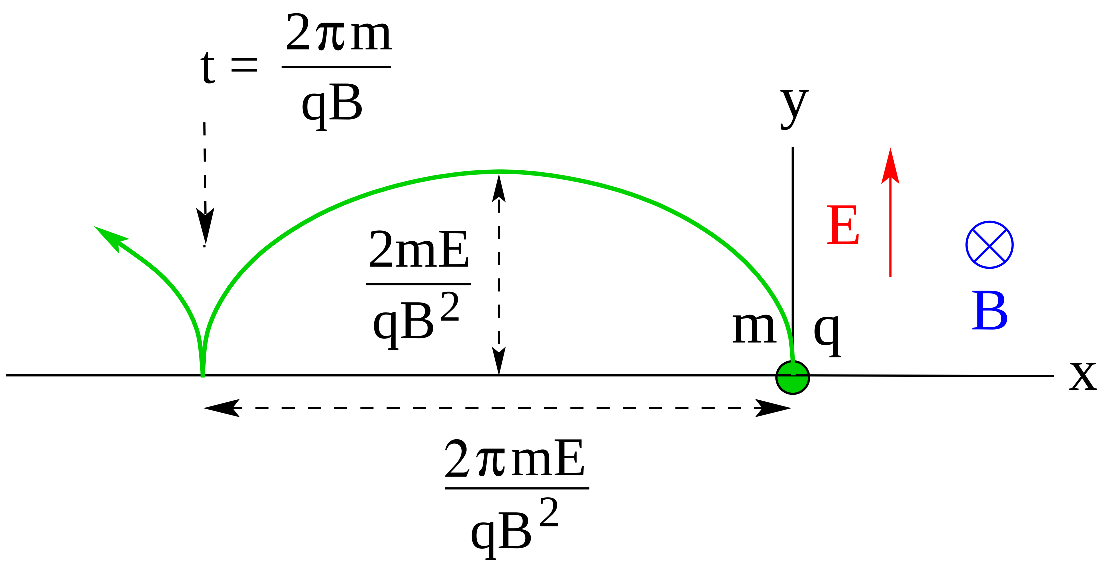
$$v_x(t) = \frac{E}{B}\left[\cos\left(\frac{qBt}{m}\right) - 1\right], \quad v_y(t) = \frac{E}{B}\sin\left(\frac{qBt}{m}\right)$$

- Solution for position of particle:

$$x(t) = \frac{E}{B}\int_0^t \left[\cos\left(\frac{qBt}{m}\right) - 1\right] dt = \frac{Em}{qB^2}\sin\left(\frac{qBt}{m}\right) - \frac{Et}{B}$$

$$y(t) = \frac{E}{B}\int_0^t \sin\left(\frac{qBt}{m}\right) dt = \frac{Em}{qB^2}\left[1 - \cos\left(\frac{qBt}{m}\right)\right]$$

- Path of particle in (x, y)-plane: cycloid

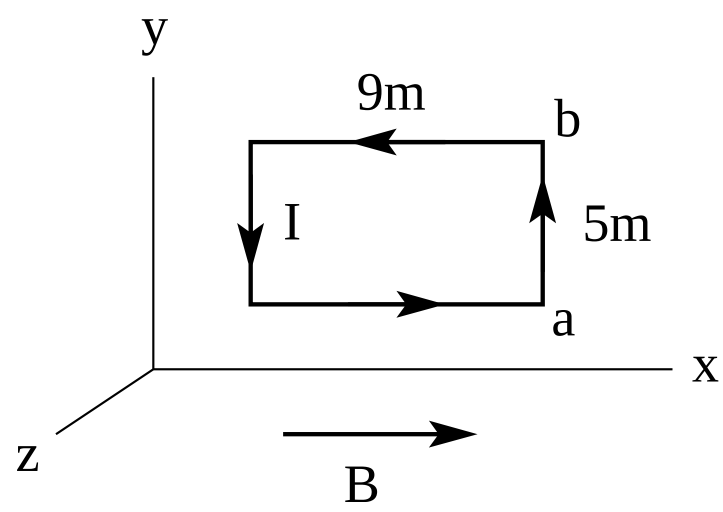


Consider a rectangular conducting loop in the xy-plane with a counterclockwise current $I = 7$A in a uniform magnetic field $\vec{B} = 3\mathrm{T}\hat{i}$.

(a) Find the magnetic moment $\vec{\mu}$ (magnitude and direction) of the loop.

(b) Find the force $\vec{F}$ (magnitude and direction) acting on the side ab of the rectangle.

(c) Find the torque $\vec{\tau}$ (magnitude and direction) acting on the loop.



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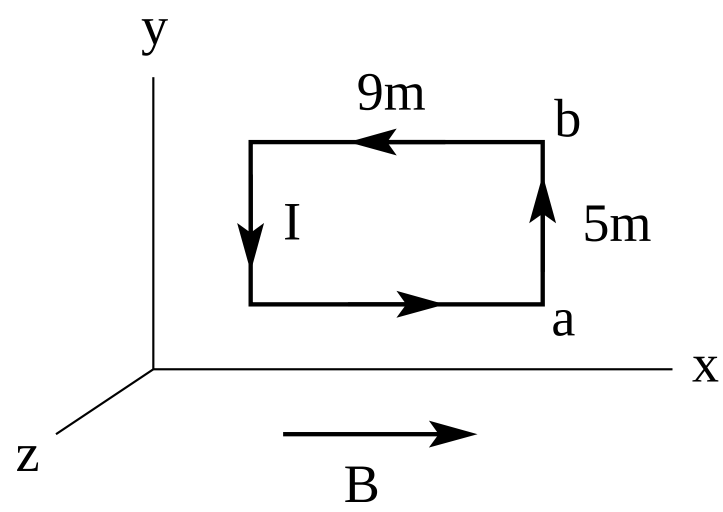

Solution:

(a) $\vec{\mu} = (7\mathrm{A})(45\mathrm{m}^2)\hat{k} = 315\mathrm{Am}^2\hat{k}.$

Consider a rectangular conducting loop in the xy-plane with a counterclockwise current $I = 7$A in a uniform magnetic field $\vec{B} = 3\mathrm{T}\hat{i}$.
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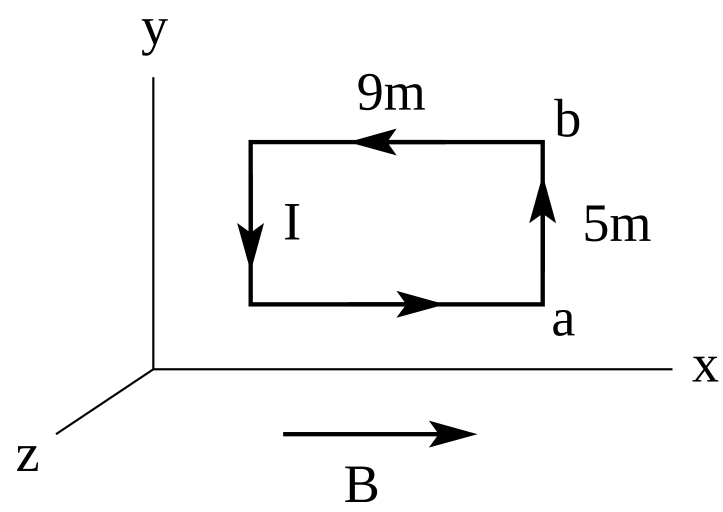

Solution:

(a) $\vec{\mu} = (7\mathrm{A})(45\mathrm{m}^2)\hat{k} = 315\mathrm{Am}^2\hat{k}$.

(b) $\vec{F} = I\vec{L} \times \vec{B} = (7\mathrm{A})(5\mathrm{m}\hat{j}) \times (3\mathrm{T}\hat{i}) = -105\mathrm{N}\hat{k}$.

Consider a rectangular conducting loop in the xy-plane with a counterclockwise current $I = 7$A in a uniform magnetic field $\vec{B} = 3\mathrm{T}\hat{i}$.

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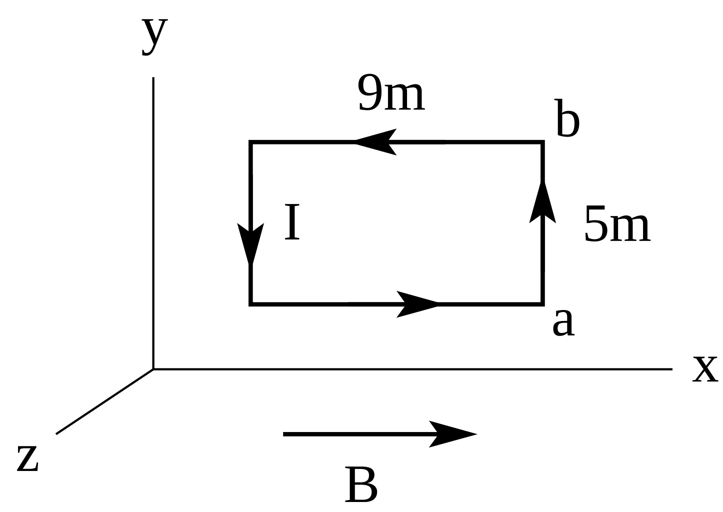

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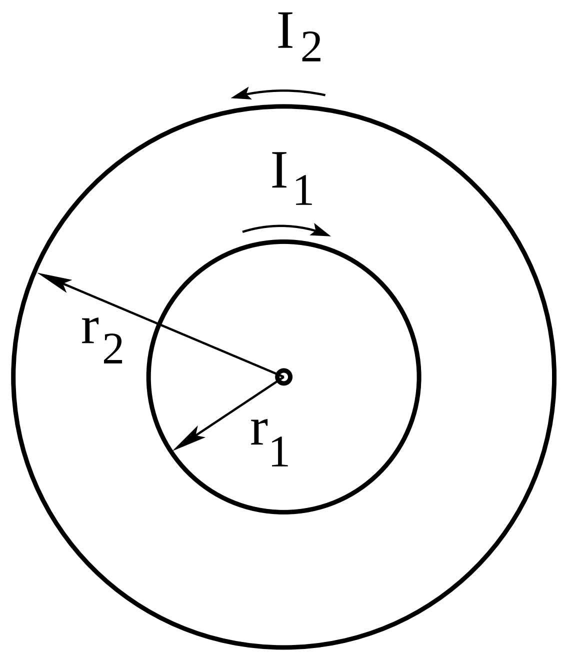
(c) $\vec{\tau} = \vec{\mu} \times \vec{B} = (315\mathrm{Am}^2\hat{k}) \times (3\mathrm{T}\hat{i}) = 945\mathrm{Nm}\hat{j}$

Unit Exam III: Problem #1 (Spring ’08)

Consider two circular currents $I_1 = 3$A at radius $r_1 = 2$m and $I_2 = 5$A at radius $r_2 = 4$m in the directions shown.
(a) Find magnitude B and direction $(\odot, \otimes)$ of the resultant magnetic field at the center.
(b) Find magnitude μ and direction $(\odot, \otimes)$ of the magnetic dipole moment generated by the two currents.



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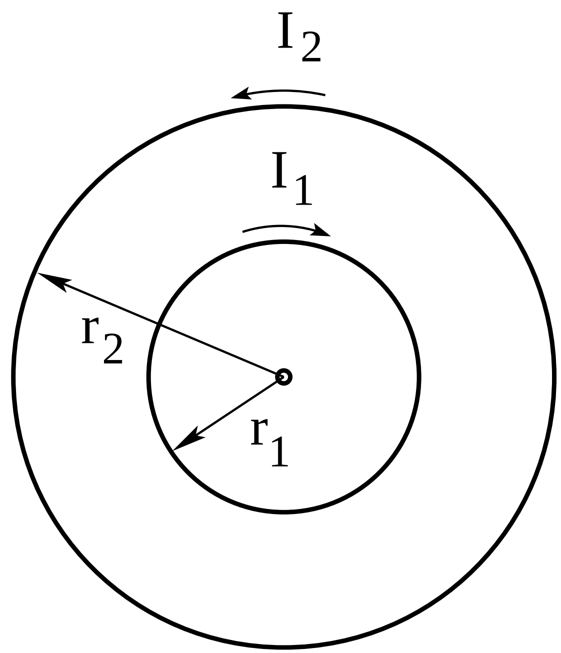

Solution:

(a) $B = \frac{\mu_0(3\mathrm{A})}{2(2\mathrm{m})} - \frac{\mu_0(5\mathrm{A})}{2(4\mathrm{m})} = (9.42 - 7.85) \times 10^{-7}\mathrm{T}$

$\Rightarrow B = 1.57 \times 10^{-7}\mathrm{T} \quad \otimes$

Unit Exam III: Problem #1 (Spring '08)

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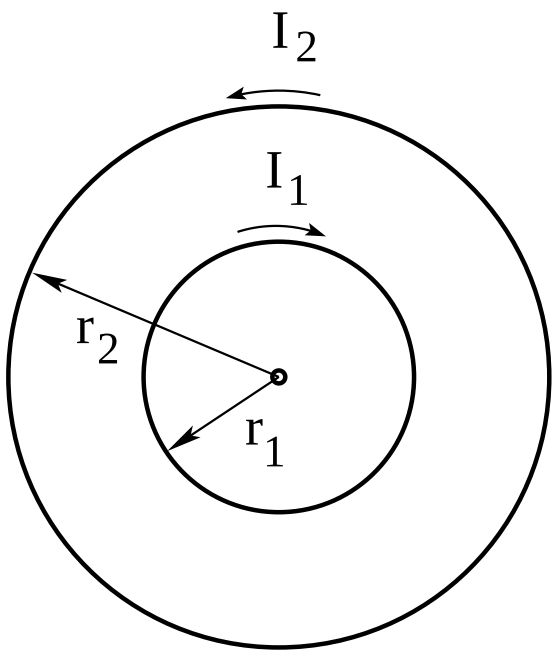

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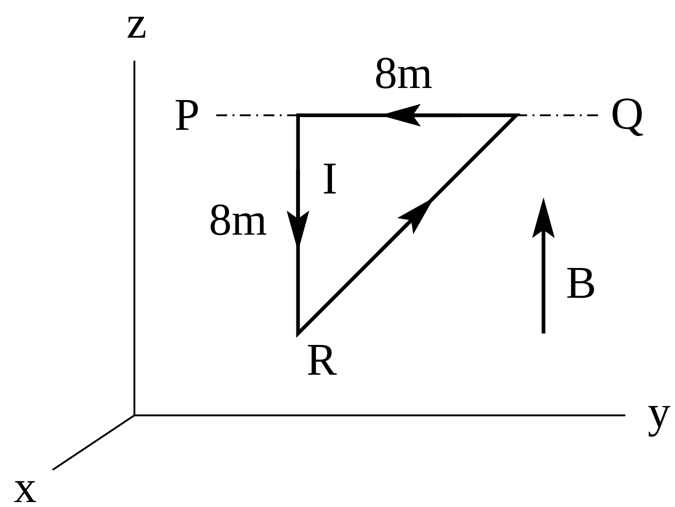
$\Rightarrow \ B = 1.57 \times 10^{-7}\mathrm{T} \quad \otimes$

(b) $\mu = \pi(4\mathrm{m})^2(5\mathrm{A}) - \pi(2\mathrm{m})^2(3\mathrm{A}) = (251 - 38)\mathrm{Am}^2$

$\Rightarrow \ \mu = 213\mathrm{Am}^2 \quad \odot$

A triangular conducting loop in the yz-plane with a counterclockwise current $I = 3$A is free to rotate about the axis PQ. A uniform magnetic field $\vec{B} = 0.5\mathrm{T}\hat{k}$ is present. (a) Find the magnetic moment $\vec{\mu}$ (magnitude and direction) of the triangle.
(b) Find the magnetic torque $\vec{\tau}$ (magnitude and direction) acting on the triangle.
(c) Find the magnetic force $\vec{F}_H$ (magnitude and direction) acting on the long side (hypotenuse) of the triangle.
(d) Find the force $\vec{F}_R$ (magnitude and direction) that must be applied to the corner R to keep the triangle from rotating.



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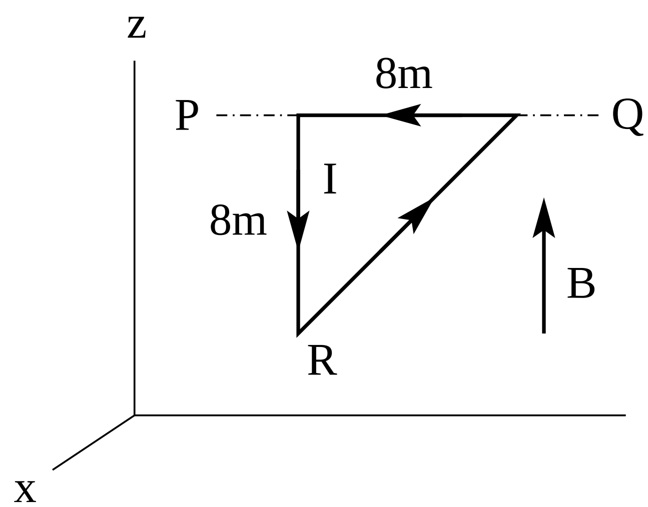


Solution:

(a) $\vec{\mu} = (3\text{A})(32\text{m}^2)\hat{i} = 96\text{Am}^2\hat{i}.$

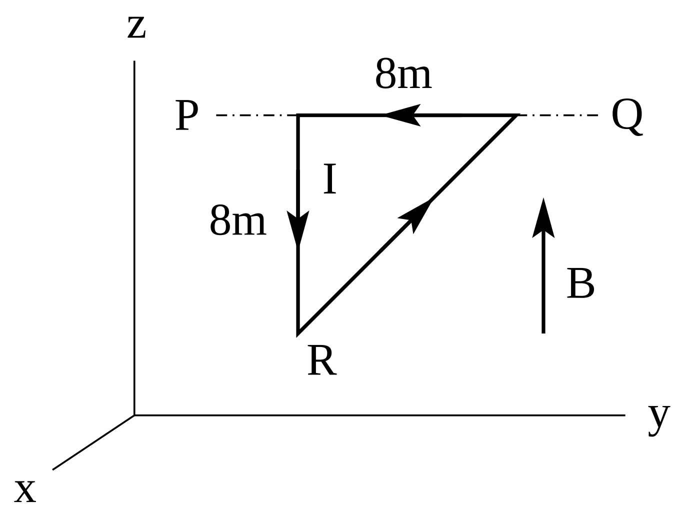
Unit Exam III: Problem #1 (Spring '09)

A triangular conducting loop in the yz-plane with a counterclockwise current $I = 3$A is free to rotate about the axis PQ. A uniform magnetic field $\vec{B} = 0.5\mathrm{T}\hat{k}$ is present. (a) Find the magnetic moment $\vec{\mu}$ (magnitude and direction) of the triangle.

(b) Find the magnetic torque $\vec{\tau}$ (magnitude and direction) acting on the triangle.

(c) Find the magnetic force $\vec{F}_H$ (magnitude and direction) acting on the long side (hypotenuse) of the triangle.

(d) Find the force $\vec{F}_R$ (magnitude and direction) that must be applied to the corner R to keep the triangle from rotating.



Solution:

(a) $\vec{\mu} = (3\mathrm{A})(32\mathrm{m}^2)\hat{i} = 96\mathrm{Am}^2\hat{i}$.

(b) $\vec{\tau} = \vec{\mu} \times \vec{B} = (96\mathrm{Am}^2\hat{i}) \times (0.5\mathrm{T}\hat{k}) = -48\mathrm{Nm}\hat{j}$.

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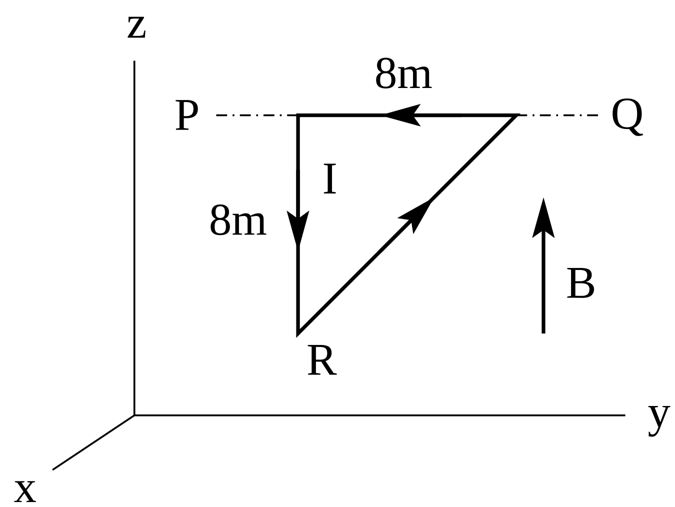

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(c) $F_H = (3\text{A})(8\sqrt{2}\text{m})(0.5\text{T})(\sin 45^\circ) = 12\text{N} \quad \odot$.

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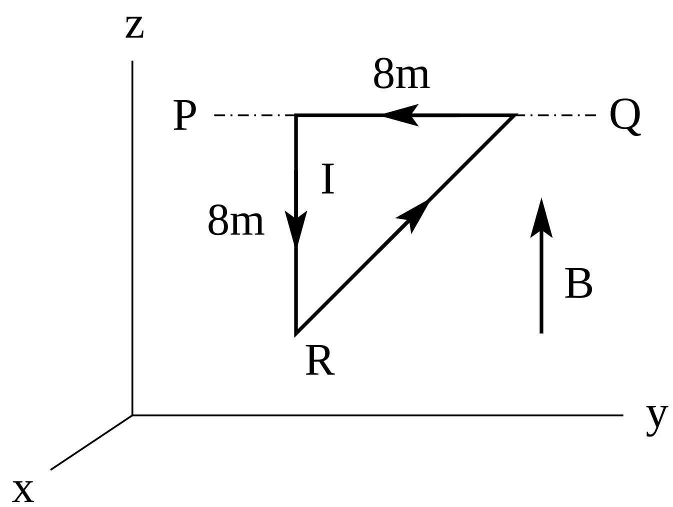

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(c) $F_H = (3\mathrm{A})(8\sqrt{2}\mathrm{m})(0.5\mathrm{T})(\sin 45^\circ) = 12\mathrm{N} \quad \odot$.

(d) $(-8\mathrm{m}\hat{k}) \times \vec{F}_R = -\vec{\tau} = 48\mathrm{Nm}\hat{j} \quad \Rightarrow \quad \vec{F}_R = -6\mathrm{N}\hat{i}$.