Mechanical Oscillator

- law of motion: \( F = ma \), \( a = \frac{d^2 x}{dt^2} \)
- law of force: \( F = -kx \)
- equation of motion: \( \frac{d^2 x}{dt^2} = -\frac{k}{m} x \)
- displacement: \( x(t) = x_{max} \cos(\omega t) \)
- velocity: \( v(t) = -\omega x_{max} \sin(\omega t) \)
- angular frequency: \( \omega = \sqrt{\frac{k}{m}} \)
- kinetic energy: \( K = \frac{1}{2} mv^2 \)
- potential energy: \( U = \frac{1}{2} kx^2 \)
- total energy: \( E = K + U = \text{const.} \)
Electromagnetic Oscillator (LC Circuit)

- loop rule: \( \frac{Q}{C} + L \frac{dI}{dt} = 0 \), \( I = \frac{dQ}{dt} \)

- equation of motion: \( \frac{d^2Q}{dt^2} = -\frac{1}{LC}Q \)

- charge on capacitor: \( Q(t) = Q_{\text{max}} \cos(\omega t) \)

- current through inductor: \( I(t) = -\omega Q_{\text{max}} \sin(\omega t) \)

- angular frequency: \( \omega = \frac{1}{\sqrt{LC}} \)

- magnetic energy: \( U_B = \frac{1}{2} LI^2 \) (stored on inductor)

- electric energy: \( U_E = \frac{Q^2}{2C} \) (stored on capacitor)

- total energy: \( E = U_B + U_E = \text{const.} \)
Mechanical vs Electromagnetic Oscillations

mechanical oscillations

- position: $x(t) = A \cos(\omega t)$ [red]
- velocity: $v(t) = -A \sin(\omega t)$ [green]
- period: $\tau = \frac{2\pi}{\omega}$, $\omega = \sqrt{\frac{k}{m}}$

![](image1)

electromagnetic oscillations

- charge: $Q(t) = A \cos(\omega t)$ [red]
- current: $I(t) = -A \sin(\omega t)$ [green]
- period: $\tau = \frac{2\pi}{\omega}$, $\omega = \frac{1}{\sqrt{LC}}$

![](image2)

potential energy: $U(t) = \frac{1}{2} k x^2(t)$ [r]

kinetic energy: $K(t) = \frac{1}{2} m v^2(t)$ [g]

total energy: $E = U(t) + K(t) = \text{const}$

electric energy: $U_E(t) = \frac{1}{2C} Q^2(t)$ [r]

magnetic energy: $U_B(t) = \frac{1}{2} LI^2(t)$ [g]

total energy: $E = U_E(t) + U_B(t) = \text{const}$
Mechanical Oscillator with Damping

- Law of motion: \( F = ma, \quad a = \frac{d^2 x}{dt^2} \)
- Law of force: \( F = -kx - bv, \quad v = \frac{dx}{dt} \)
- Equation of motion: \( \frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \)

Solution for initial conditions \( x(0) = A, \ v(0) = 0 \):

(a) **underdamped motion**: \( b^2 < 4km \)

\[
x(t) = Ae^{-bt/2m} \left[ \cos(\omega't) + \frac{b}{2m\omega'} \sin(\omega't) \right] \quad \text{with} \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}
\]

(b) **overdamped motion**: \( b^2 > 4km \)

\[
x(t) = Ae^{-bt/2m} \left[ \cosh(\Omega't) + \frac{b}{2m\Omega'} \sinh(\Omega't) \right] \quad \text{with} \quad \Omega' = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}
\]
Damped Electromagnetic Oscillator (RLC Circuit)

- loop rule: \(RI + L \frac{dI}{dt} + \frac{Q}{C} = 0, \quad I = \frac{dQ}{dt}\)
- equation of motion: \(\frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{LC}Q = 0\)

Solution for initial conditions \(Q(0) = Q_{max}, \ I(0) = 0\):

(a) underdamped motion: \(R^2 < \frac{4L}{C}\)

\[
Q(t) = Q_{max} e^{-Rt/2L} \left[ \cos(\omega't) + \frac{R}{2L\omega'} \sin(\omega't) \right]
\]
with \(\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\)

(b) overdamped motion: \(R^2 > \frac{4L}{C}\)

\[
Q(t) = Q_{max} e^{-Rt/2L} \left[ \cosh(\Omega't) + \frac{R}{2L\Omega'} \sinh(\Omega't) \right]
\]
with \(\Omega' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\)
Name the \( LC \) circuit with the highest and the lowest angular frequency of oscillation.
At time $t = 0$ a charge $Q = 2C$ is on each capacitor and all currents are zero.

(a) What is the energy stored in the circuit?
(b) At what time $t_1$ are the capacitors discharged for the first time?
(c) What is the current through each inductor at time $t_1$?
In these $LC$ circuits all capacitors have equal capacitance $C$ and all inductors have equal inductance $L$. Sort the circuits into groups that are equivalent.
Oscillator with Two Modes

Electromagnetic:

mode #1:  \[ L \frac{dI}{dt} + \frac{Q}{C} + \frac{Q}{C} + L \frac{dI}{dt} = 0, \quad I = \frac{dQ}{dt} \]

\[ \Rightarrow \frac{dI}{dt} = -\frac{Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \frac{1}{\sqrt{LC}} \]

mode #2:  \[ L \frac{dI}{dt} + \frac{Q}{C} + \frac{2Q}{C} = 0, \quad I = \frac{dQ}{dt} \]

\[ \Rightarrow \frac{dI}{dt} = -\frac{3Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \sqrt{\frac{3}{LC}} \]

Mechanical:

mode #1:  \[ \omega = \sqrt{\frac{k}{m}} \]

mode #2:  \[ \omega = \sqrt{\frac{3k}{m}} \]
In the circuit shown the capacitor is without charge. When the switch is closed to position $a$...

(a) find the initial rate $dI/dt$ at which the current increases from zero,
(b) find the charge $Q$ on the capacitor after a long time.

Then, when the switch is thrown from $a$ to $b$...

(c) find the time $t_1$ it takes the capacitor to fully discharge,
(d) find the maximum current $I_{max}$ in the process of discharging.
In the circuit shown the capacitor is without charge and the switch is in position \textit{a}.

(i) When the switch is moved to position \textit{b} we have an \textit{RL} circuit with the current building up gradually: \( I(t) = (\mathcal{E}/R)[1 - e^{-t/\tau}] \).

Find the time constant \( \tau \) and the current \( I_{\text{max}} \) after a long time.

(ii) Then we reset the clock and move the switch from \textit{b} to \textit{c} with no interruption of the current through the inductor. We now have a \textit{LC} circuit: \( I(t) = I_{\text{max}} \cos(\omega t) \).

Find the angular frequency of oscillation \( \omega \) and the maximum charge \( Q_{\text{max}} \) that goes onto the capacitor periodically.

\[
\begin{align*}
&34\text{V} & 14\Omega \\
&6.2\mu\text{F} & 54\text{mH}
\end{align*}
\]
In the circuit shown the capacitor is without charge and the switch is in position \( a \).

(i) When the switch is moved to position \( b \) we have an \( RC \) circuit with the capacitor being charged up gradually: 
\[
Q(t) = \mathcal{E}C[1 - e^{-t/\tau}].
\]
Find the time constant \( \tau \) and the charge \( Q_{\text{max}} \) after a long time.

(ii) Then we reset the clock and move the switch from \( b \) to \( c \).
We now have a \( LC \) circuit: 
\[
Q(t) = Q_{\text{max}} \cos(\omega t).
\]
Find the angular frequency of oscillation \( \omega \) and the maximum current \( I_{\text{max}} \) that flows through the inductor periodically.