Capacitor (device):

- Two oppositely charged conductors separated by an insulator.
- The charges $+Q$ and $-Q$ on conductors generate an electric field $\vec{E}$ and a potential difference $V$ (voltage).
- Only one conductor may be present. Then the relevant potential difference is between the conductor and a point at infinity.

Capacitance (device property):

- Definition: $C = \frac{Q}{V}$
- SI unit: 1F = 1C/V (one Farad)
**Parallel-Plate Capacitor**

- $A$: area of each plate
- $d$: distance between plates
- $Q$: magnitude of charge on inside surface of each plate

Charge per unit area (magnitude) on each plate: $\sigma = \frac{Q}{A}$

Uniform electric field between plates:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

Voltage between plates:

$$V \equiv V_+ - V_- = Ed = \frac{Qd}{\varepsilon_0 A}$$

Capacitance for parallel-plate geometry:

$$C \equiv \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$$
Cylindrical Capacitor

Conducting cylinder of radius $a$ and length $L$ surrounded concentrically by conducting cylindrical shell of inner radius $b$ and equal length.

- Assumption: $L \gg b$.
- $\lambda$: charge per unit length (magnitude) on each cylinder
- $Q = \lambda L$: magnitude of charge on each cylinder
- Electric field between cylinders: use Gauss’ law
  \[ E[2\pi rL] = \frac{\lambda L}{\epsilon_0} \Rightarrow E(r) = \frac{\lambda}{2\pi \epsilon_0 r} \]
- Electric potential between cylinders: use $V(a) = 0$
  \[ V(r) = -\int_a^r E(r)dr = -\frac{\lambda}{2\pi \epsilon_0} \int_a^r \frac{dr}{r} = -\frac{\lambda}{2\pi \epsilon_0} \ln \frac{r}{a} \]
- Voltage between cylinders:
  \[ V \equiv V_+ - V_- = V(a) - V(b) = \frac{Q}{2\pi \epsilon_0 L} \ln \frac{b}{a} \]
- Capacitance for cylindrical geometry:
  \[ C \equiv \frac{Q}{V} = \frac{2\pi \epsilon_0 L}{\ln(b/a)} \]
Spherical Capacitor

Conducting sphere of radius $a$ surrounded concentrically by conducting spherical shell of inner radius $b$.

- $Q$: magnitude of charge on each sphere
- Electric field between spheres: use Gauss’ law
  \[ E[4\pi r^2] = \frac{Q}{\varepsilon_0} \Rightarrow E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \]
- Electric potential between spheres: use $V(a) = 0$
  \[ V(r) = -\int_{a}^{r} E(r)\,dr = -\frac{Q}{4\pi\varepsilon_0} \int_{a}^{r} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0} \left[ \frac{1}{r} - \frac{1}{a} \right] \]
- Voltage between spheres:
  \[ V \equiv V_+ - V_- = V(a) - V(b) = \frac{Q}{4\pi\varepsilon_0} \frac{b - a}{ab} \]
- Capacitance for spherical geometry:
  \[ C \equiv \frac{Q}{V} = 4\pi\varepsilon_0 \frac{ab}{b - a} \]
Energy Stored in Capacitor

Charging a capacitor requires work. The work done is equal to the potential energy stored in the capacitor.

While charging, $V$ increases linearly with $q$:

$$V(q) = \frac{q}{C}.$$ 

Increment of potential energy:

$$dU = Vdq = \frac{q}{C}dq.$$ 

Potential energy of charged capacitor:

$$U = \int_0^Q Vdq = \frac{1}{C} \int_0^Q qdq = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV.$$ 

Q: where is the potential energy stored?
A: in the electric field.
Energy is stored in the electric field between the plates of a capacitor.

- Capacitance: \( C = \frac{\epsilon_0 A}{d} \).
- Voltage: \( V = Ed \).
- Potential energy: \( U = \frac{1}{2} CV^2 = \frac{1}{2} \epsilon_0 E^2 (Ad) \).
- Volume between the plates: \( Ad \).
- Energy density of the electric field: \( u_E = \frac{U}{Ad} = \frac{1}{2} \epsilon_0 E^2 \)
Integrating Energy Density in Spherical Capacitor

- Electric field: $E(r) = \frac{Q}{4\pi \varepsilon_0} \frac{1}{r^2}$

- Voltage: $V = \frac{Q}{4\pi \varepsilon_0} \frac{b - a}{ab} = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{a} - \frac{1}{b} \right]$  

- Energy density: $u_E(r) = \frac{1}{2} \varepsilon_0 E^2(r)$

- Energy stored in capacitor: $U = \int_a^b u_E(r)(4\pi r^2)dr$

- $\Rightarrow U = \int_a^b \frac{1}{2} \frac{Q^2}{(4\pi \varepsilon_0)^2} \frac{1}{r^4} (4\pi r^2)dr$

- $\Rightarrow U = \frac{1}{2} \frac{Q^2}{4\pi \varepsilon_0} \int_a^b \frac{1}{r^2} dr = \frac{1}{2} \frac{Q^2}{4\pi \varepsilon_0} \left[ \frac{1}{a} - \frac{1}{b} \right] = \frac{1}{2} QV$
Capacitor Problem (1)

Consider two oppositely charged parallel plates separated by a very small distance $d$.

What happens when the plates are pulled apart a fraction of $d$? Will the quantities listed below increase or decrease in magnitude or stay unchanged?

(a) Electric field $\vec{E}$ between the plates.
(b) Voltage $V$ across the plates.
(c) Capacitance $C$ of the device.
(d) Energy $U$ stored in the device.
Find the equivalent capacitance of two capacitors connected in parallel:

- Charge on capacitors: $Q_1 + Q_2 = Q$
- Voltage across capacitors: $V_1 = V_2 = V$
- Equivalent capacitance:
  \[ C = \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V_1} + \frac{Q_2}{V_2} \]
- $\Rightarrow C = C_1 + C_2$
Capacitors Connected in Series

Find the equivalent capacitance of two capacitors connected in series:

- Charge on capacitors: \( Q_1 = Q_2 = Q \)
- Voltage across capacitors: \( V_1 + V_2 = V \)
- Equivalent capacitance:
  \[
  \frac{1}{C} \equiv \frac{V}{Q} = \frac{V_1 + V_2}{Q} = \frac{V_1}{Q_1} + \frac{V_2}{Q_2}
  \]
  \[
  \Rightarrow \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}
  \]
Consider two equal capacitors connected in series.

(a) Find the voltages \( V_A - V_B, V_B - V_C, V_A - V_D \).
(b) Find the charge \( Q_A \) on plate \( A \).
(c) Find the electric field \( E \) between plates \( C \) and \( D \).
Find the equivalent capacitances of the two capacitor networks. All capacitors have a capacitance of $1 \mu F$. 

(a) 

(b)
Consider the two capacitors connected in parallel.

(a) Which capacitor has the higher voltage?
(b) Which capacitor has more charge?
(c) Which capacitor has more energy?

Consider the two capacitors connected in series.

(d) Which capacitor has the higher voltage?
(e) Which capacitor has more charge?
(f) Which capacitor has more energy?
Connect the three capacitors in such a way that the equivalent capacitance is $C_{eq} = 4 \mu F$. Draw the circuit diagram.

\[ \begin{array}{c}
2 \mu F \\
\hline
\end{array} \quad \begin{array}{c}
2 \mu F \\
\hline
\end{array} \quad \begin{array}{c}
3 \mu F \\
\hline
\end{array} \quad \begin{array}{c}
4 \mu F \\
\hline
\end{array} \]
Connect the three capacitors in such a way that the equivalent capacitance is $C_{eq} = 2 \mu F$. Draw the circuit diagram.

$$\begin{align*}
1 \mu F & \quad 3 \mu F & \quad 5 \mu F \\
\quad & \quad & \\
\quad & \quad & \\
\quad & \quad & \\
\quad & \quad & \\
2 \mu F & \\
\end{align*}$$
Find the equivalent capacitances of the following circuits.

(a)  

(b)
(a) Name two capacitors from the circuit on the left that are connected in series.
(b) Name two capacitors from the circuit on the right that are connected in parallel.
(a) In the circuit shown the switch is first thrown to A. Find the charge $Q_0$ and the energy $U_A$ on the capacitor $C_1$ once it is charged up.

(b) Then the switch is thrown to $B$, which charges up the capacitors $C_2$ and $C_3$. The capacitor $C_1$ is partially discharged in the process. Find the charges $Q_1, Q_2, Q_3$ on all three capacitors and the voltages $V_1, V_2, V_3$ across each capacitor once equilibrium has been reached again. What is the energy $U_B$ now stored in the circuit?

\[ C_1 = 4\, \mu F \]
\[ C_2 = 3\, \mu F \]
\[ C_3 = 6\, \mu F \]
\[ V_0 = 120\, V \]
In the circuit shown find the charges $Q_1, Q_2, Q_3, Q_4$ on each capacitor and the voltages $V_1, V_2, V_3, V_4$ across each capacitor

(a) when the switch $S$ is open,
(b) when the switch $S$ is closed.
The circuit of capacitors connected to a battery is at equilibrium.

(a) Find the equivalent capacitance $C_{eq}$.

(b) Find the total energy $U$ stored in the circuit (excluding the battery).

(c) Find the the charge $Q_3$ on capacitor $C_3$.

(d) Find the voltage $V_2$ across capacitor $C_2$. 

![Capacitor Circuit Diagram]

- $C_1 = 3 \mu F$
- $C_3 = 4 \mu F$
- $C_2 = 6 \mu F$
- $12 V$
No two capacitors are in parallel or in series. Solution requires different strategy:

- zero charge on each conductor (here color coded),
- zero voltage around any closed loop.

Specifications: \( C_1, \ldots, Q_5, V \).

Five equations for unknowns \( Q_1, \ldots, Q_5 \):

- \( Q_1 + Q_2 - Q_4 - Q_5 = 0 \)
- \( Q_3 + Q_4 - Q_1 = 0 \)
- \( \frac{Q_5}{C_5} + \frac{Q_3}{C_3} - \frac{Q_4}{C_4} = 0 \)
- \( \frac{Q_2}{C_2} - \frac{Q_1}{C_1} - \frac{Q_3}{C_3} = 0 \)
- \( V - \frac{Q_1}{C_1} - \frac{Q_4}{C_4} = 0 \)

Equivalent capacitance: \( C_{eq} = \frac{Q_1 + Q_2}{V} \)

(a) \( C_m = 1 \text{pF}, \ m = 1, \ldots, 5 \) and \( V = 1 \text{V} \):

\[
C_{eq} = 1 \text{pF}, \ Q_3 = 0, \quad Q_1 = Q_2 = Q_4 = Q_5 = \frac{1}{2} \text{pC}.
\]

(b) \( C_m = m \text{pF}, \ m = 1, \ldots, 5 \) and \( V = 1 \text{V} \):

\[
C_{eq} = \frac{159}{71} \text{pF}, \ Q_1 = \frac{55}{71} \text{pC}, \ Q_2 = \frac{104}{71} \text{pC}, \ Q_3 = -\frac{9}{71} \text{pC}, \ Q_4 = \frac{64}{71} \text{pC}, \ Q_5 = \frac{95}{71} \text{pC}.
\]