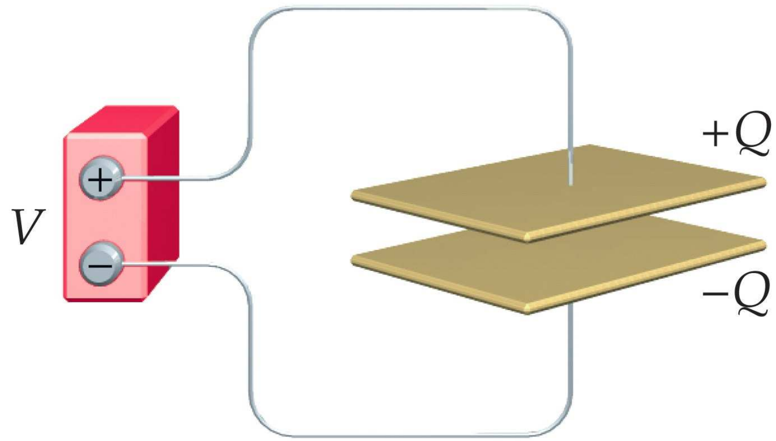
Capacitor and Capacitance

Capacitor (device):

- Two oppositely charged conductors separated by an insulator.
- The charges $+Q$ and $-Q$ on conductors generate an electric field $\vec{E}$ and a potential difference V (voltage).
- Only one conductor may be present. Then the relevant potential difference is between the conductor and a point at infinity.

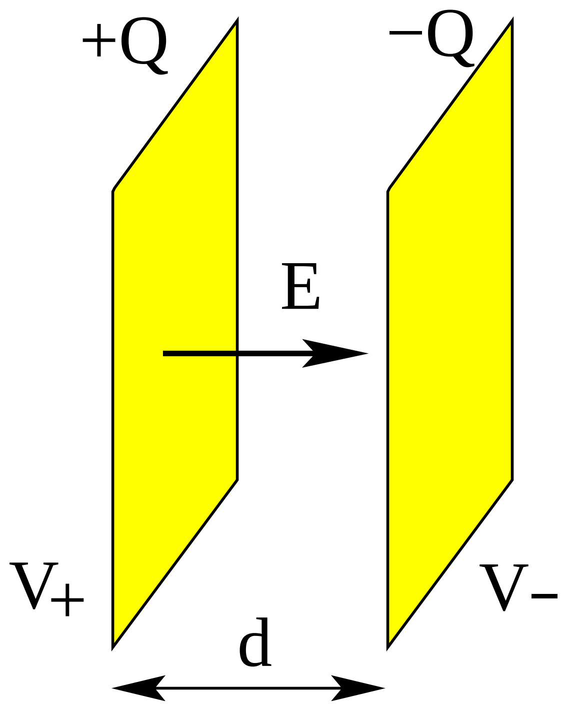
Capacitance (device property):

- Definition: $C = \dfrac{Q}{V}$
- SI unit: 1F = 1C/V (one Farad)



Parallel-Plate Capacitor

- A: area of each plate
- d: distance between plates
- Q: magnitude of charge on inside surface of each plate
- Charge per unit area (magnitude) on each plate: $\sigma = \frac{Q}{A}$
- Uniform electric field between plates:
  $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$
- Voltage between plates:
  $V \equiv V_+ - V_- = Ed = \frac{Qd}{\epsilon_0 A}$
- Capacitance for parallel-plate geometry:
  $C \equiv \frac{Q}{V} = \frac{\epsilon_0 A}{d}$



Cylindrical Capacitor

Conducting cylinder of radius a and length L surrounded concentrically by conducting cylindrical shell of inner radius b and equal length.

- Assumption: $L \gg b$.
- λ: charge per unit length (magnitude) on each cylinder
- $Q = \lambda L$: magnitude of charge on each cylinder
- Electric field between cylinders: use Gauss' law

  $$E[2\pi r L] = \frac{\lambda L}{\epsilon_0} \;\Rightarrow\; E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$

- Electric potential between cylinders: use $V(a) = 0$

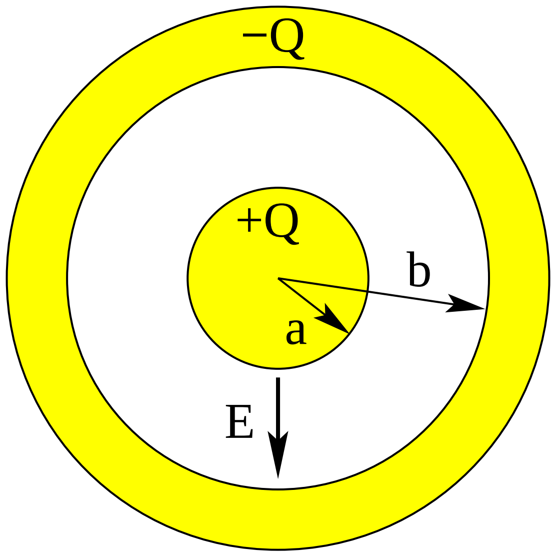
  $$V(r) = -\int_a^r E(r)dr = -\frac{\lambda}{2\pi\epsilon_0}\int_a^r \frac{dr}{r} = -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{r}{a}$$

- Voltage between cylinders:

  $$V \equiv V_+ - V_- = V(a) - V(b) = \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a}$$

- Capacitance for cylindrical geometry:

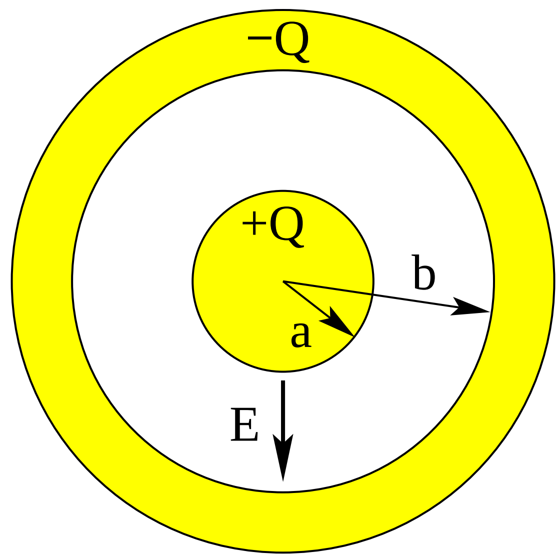
  $$C \equiv \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$$



Spherical Capacitor

Conducting sphere of radius a surrounded concentrically by conducting spherical shell of inner radius b.

- Q: magnitude of charge on each sphere
- Electric field between spheres: use Gauss' law
  $$E[4\pi r^2] = \frac{Q}{\epsilon_0} \;\Rightarrow\; E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$$
- Electric potential between spheres: use $V(a) = 0$
  $$V(r) = -\int_a^r E(r)dr = -\frac{Q}{4\pi\epsilon_0}\int_a^r \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{a}\right]$$
- Voltage between spheres:
  $$V \equiv V_+ - V_- = V(a) - V(b) = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab}$$
- Capacitance for spherical geometry:
  $$C \equiv \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}$$

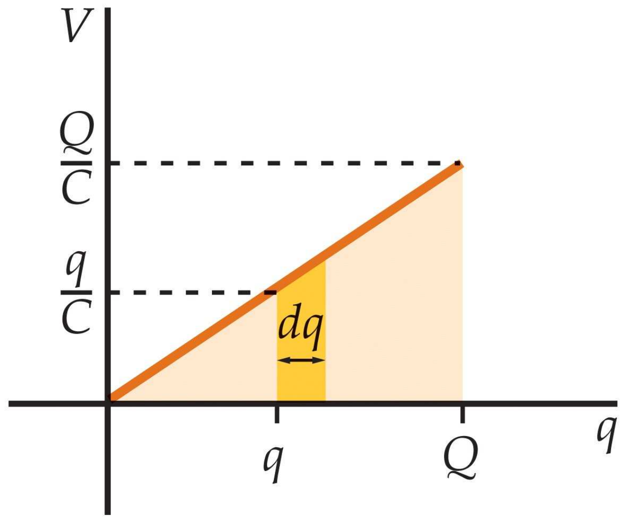


Energy Stored in Capacitor

Charging a capacitor requires work.
The work done is equal to the potential energy stored in the capacitor.

While charging, V increases linearly with q:

$$V(q) = \frac{q}{C}.$$



Increment of potential energy:

$$dU = V dq = \frac{q}{C} dq.$$

Potential energy of charged capacitor:

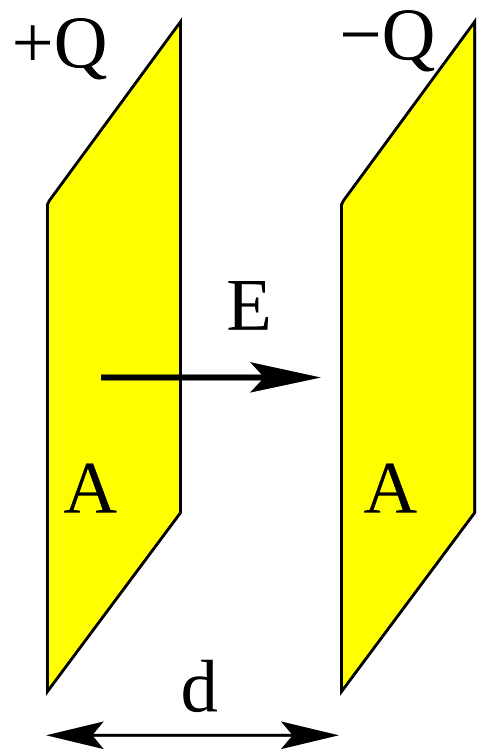
$$U = \int_0^Q V dq = \frac{1}{C}\int_0^Q q dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV.$$

Q: where is the potential energy stored?
A: in the electric field.

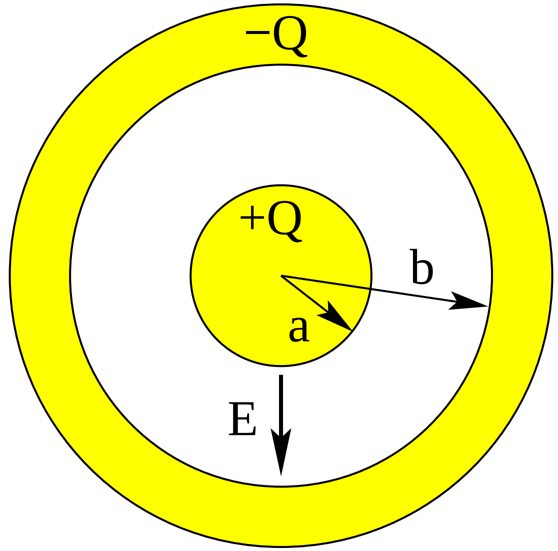
Energy Density Between Parallel Plates

Energy is stored in the electric field between the plates of a capacitor.

- Capacitance: $C = \dfrac{\epsilon_0 A}{d}$.
- Voltage: $V = Ed$.
- Potential energy: $U = \dfrac{1}{2}CV^2 = \dfrac{1}{2}\epsilon_0 E^2 (Ad)$.
- Volume between the plates: Ad.
- Energy density of the electric field: $u_E = \dfrac{U}{Ad} = \dfrac{1}{2}\epsilon_0 E^2$



Integrating Energy Density in Spherical Capacitor



- Electric field: $E(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}$
- Voltage: $V = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$
- Energy density: $u_E(r) = \frac{1}{2}\epsilon_0 E^2(r)$
- Energy stored in capacitor: $U = \int_a^b u_E(r)(4\pi r^2)dr$
- $\Rightarrow \ U = \int_a^b \frac{1}{2}\epsilon_0 \frac{Q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}(4\pi r^2)dr$
- $\Rightarrow \ U = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{1}{2}QV$

Capacitor Problem (1)

Consider two oppositely charged parallel plates separated by a very small distance d.

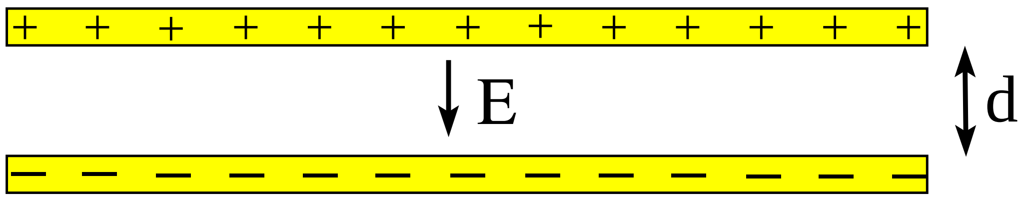
What happens when the plates are pulled apart a fraction of d? Will the quantities listed below increase or decrease in magnitude or stay unchanged?

(a) Electric field $\vec{E}$ between the plates.

(b) Voltage V across the plates.

(c) Capacitance C of the device.

(d) Energy U stored in the device.



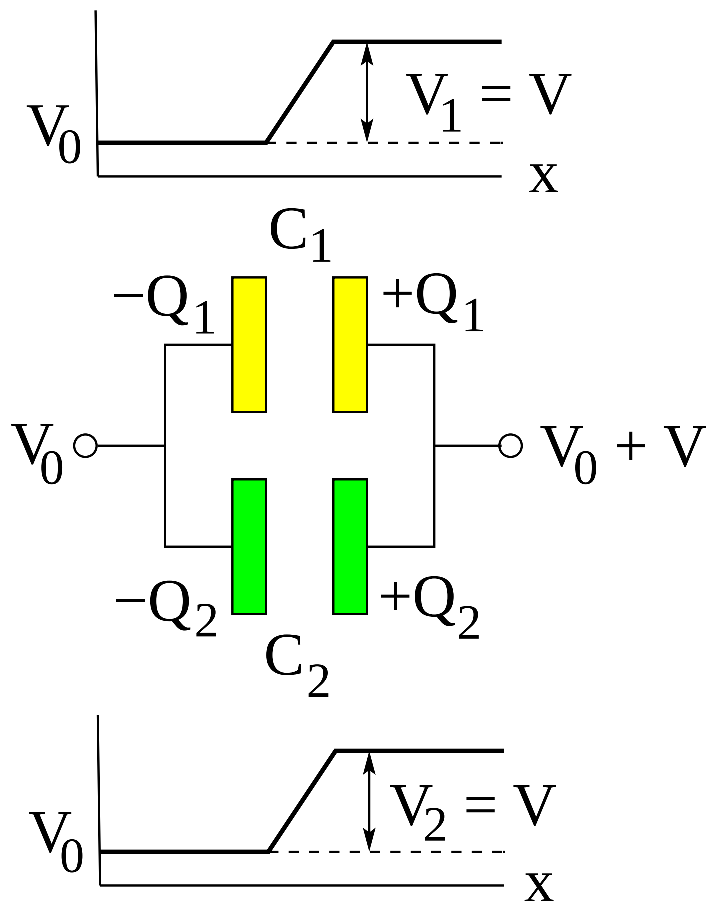
Capacitors Connected in Parallel

Find the equivalent capacitance of two capacitors connected in parallel:

- Charge on capacitors: $Q_1 + Q_2 = Q$
- Voltage across capacitors: $V_1 = V_2 = V$
- Equivalent capacitance:

$$C \equiv \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V_1} + \frac{Q_2}{V_2}$$

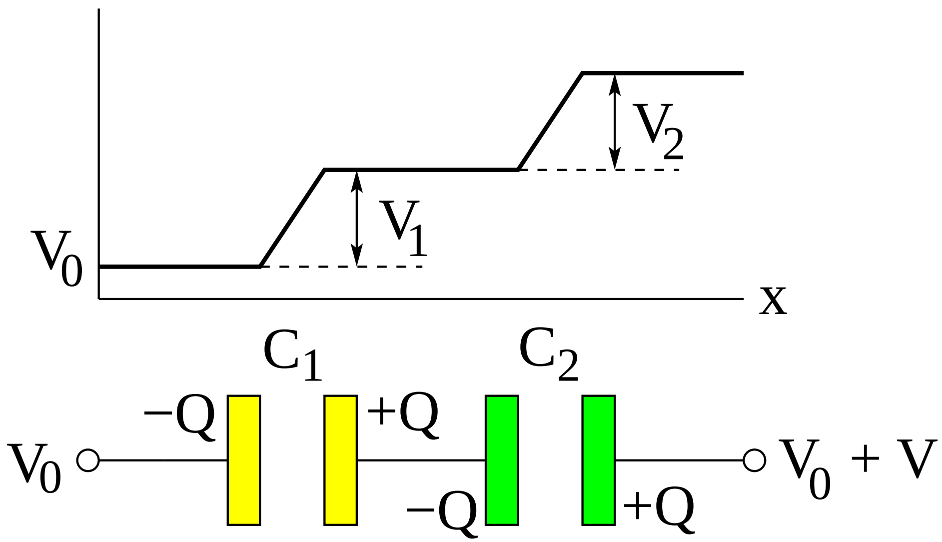
- $\Rightarrow\ C = C_1 + C_2$



Capacitors Connected in Series

Find the equivalent capacitance of two capacitors connected in series:

- Charge on capacitors: $Q_1 = Q_2 = Q$
- Voltage across capacitors: $V_1 + V_2 = V$
- Equivalent capacitance: $\frac{1}{C} \equiv \frac{V}{Q} = \frac{V_1 + V_2}{Q} = \frac{V_1}{Q_1} + \frac{V_2}{Q_2}$
- $\Rightarrow \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$



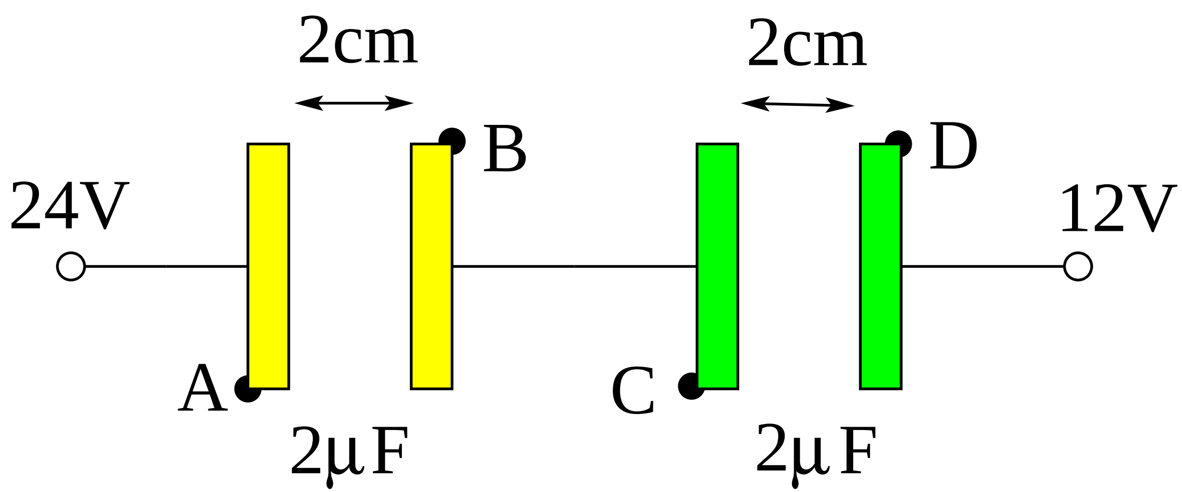
Capacitor Problem (2)

Consider two equal capacitors connected in series.

(a) Find the voltages $V_A - V_B, V_B - V_C, V_A - V_D$.

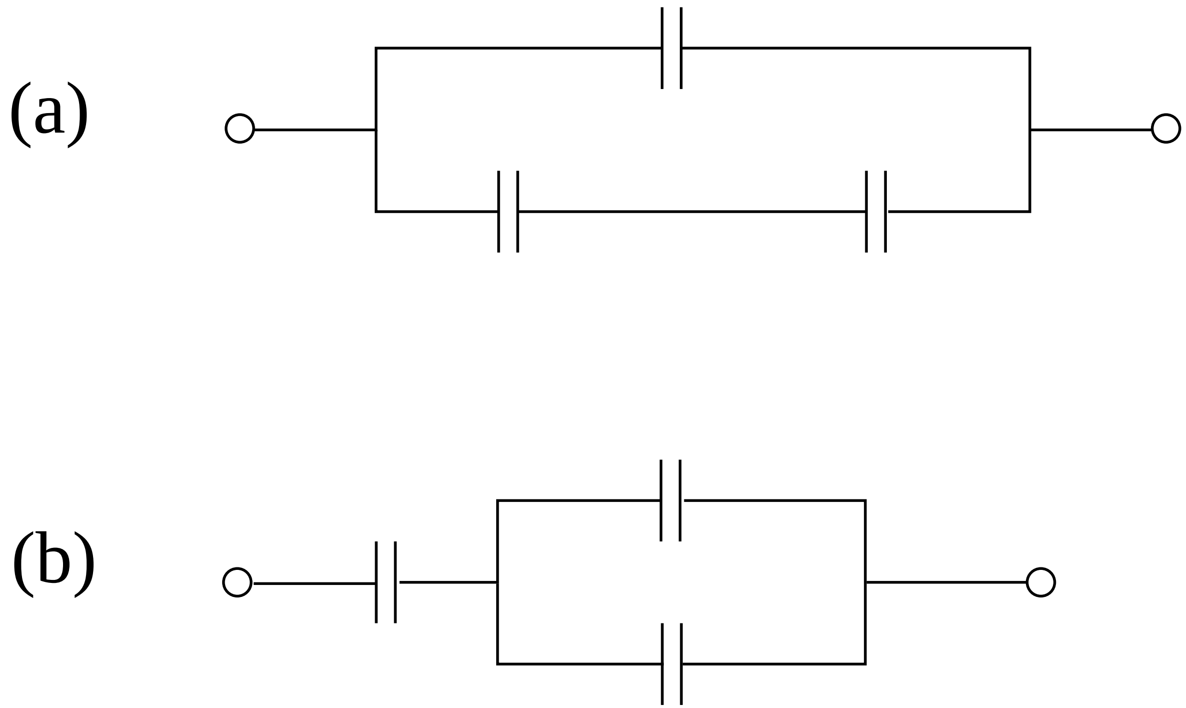
(b) Find the charge Q_A on plate A.

(c) Find the electric field E between plates C and D.



Capacitor Circuit (1)

Find the equivalent capacitances of the two capacitor networks.
All capacitors have a capacitance of $1\mu F$.

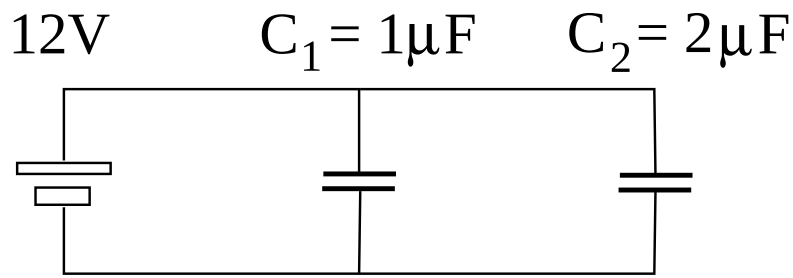


Consider the two capacitors connected in parallel.

(a) Which capacitor has the higher voltage?

(b) Which capacitor has more charge?

(c) Which capacitor has more energy?

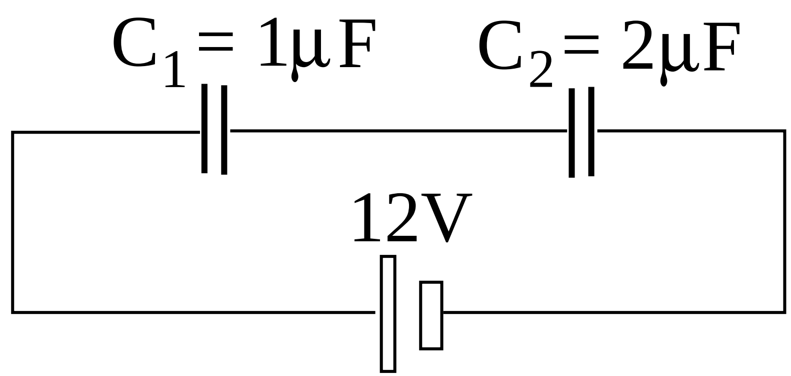


Consider the two capacitors connected in series.

(d) Which capacitor has the higher voltage?

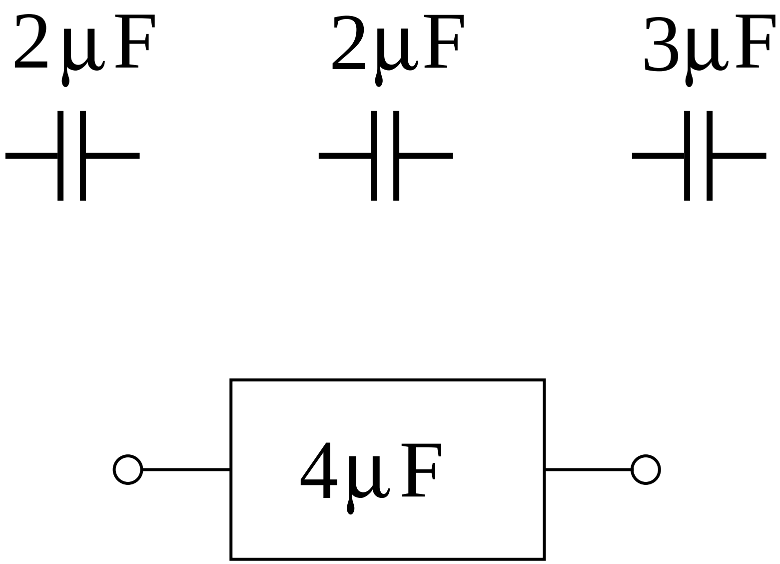
(e) Which capacitor has more charge?

(f) Which capacitor has more energy?



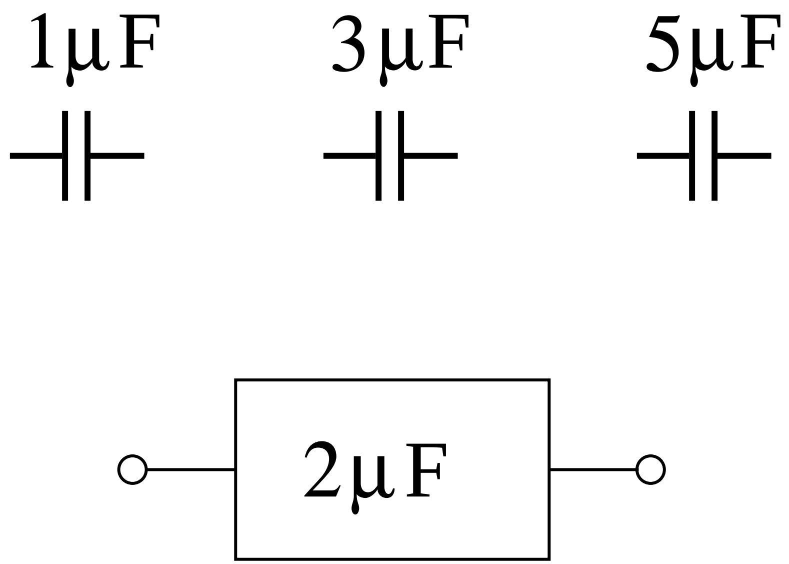
Capacitor Circuit (3)

Connect the three capacitors in such a way that the equivalent capacitance is $C_{eq} = 4\mu$F. Draw the circuit diagram.



Capacitor Circuit (4)

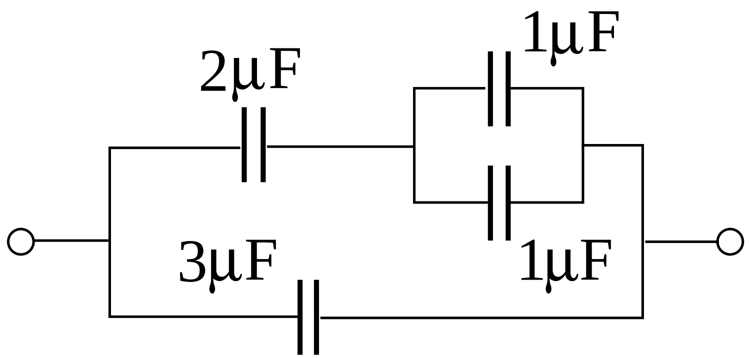
Connect the three capacitors in such a way that the equivalent capacitance is $C_{eq} = 2\mu$F. Draw the circuit diagram.



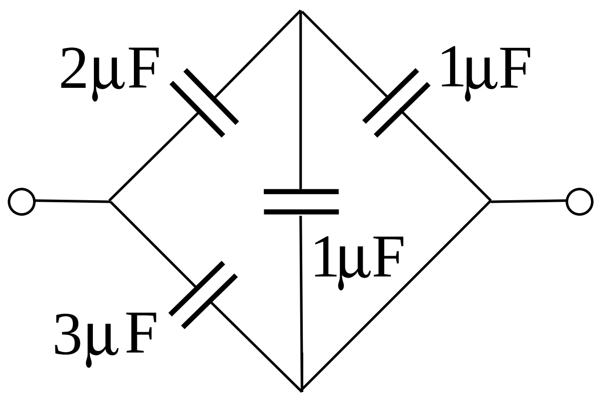
Capacitor Circuit (5)

Find the equivalent capacitances of the following circuits.

(a)



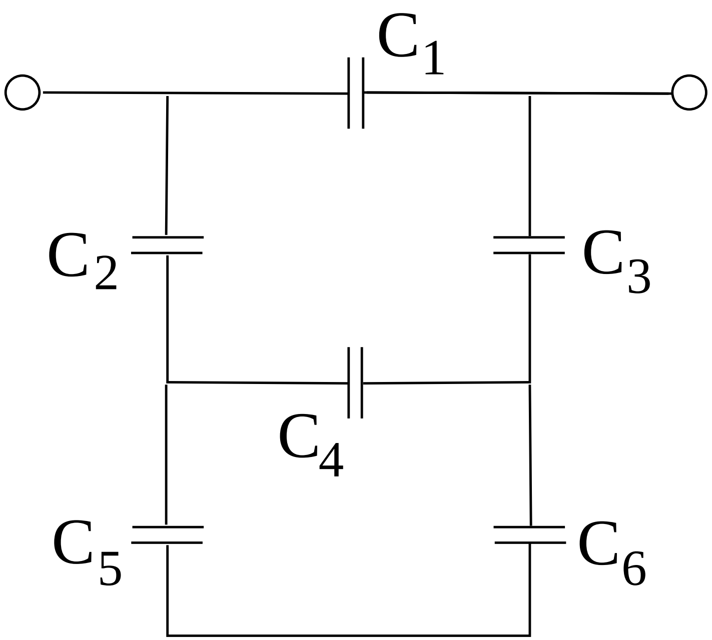
(b)

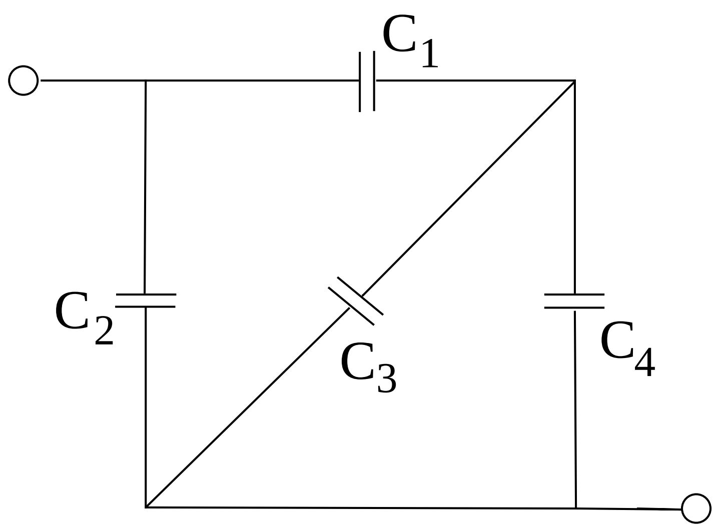


Capacitor Circuit (6)

(a) Name two capacitors from the circuit on the **left** that are connected in **series**.

(b) Name two capacitors from the circuit on the **right** that are connected in **parallel**.

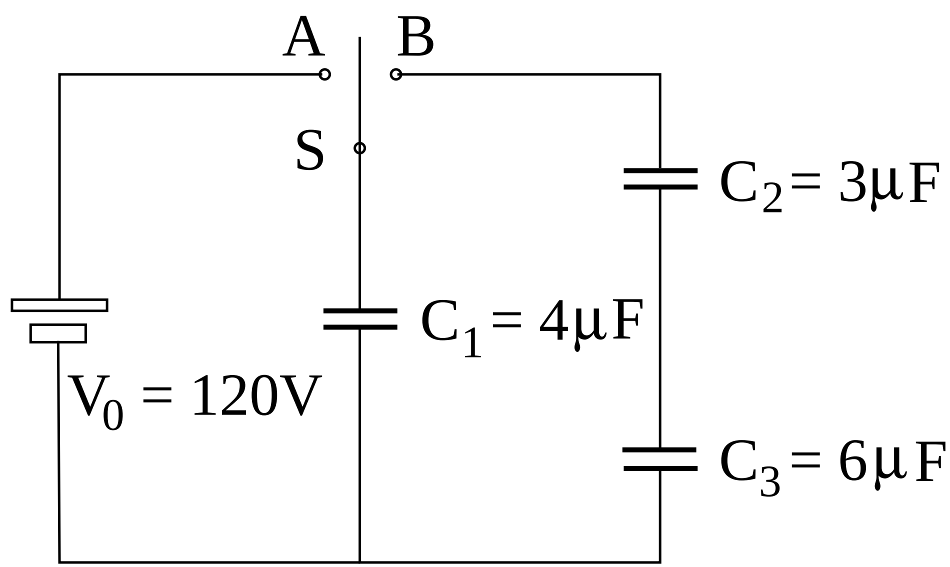




Capacitor Circuit (7)

(a) In the circuit shown the switch is first thrown to A. Find the charge Q_0 and the energy U_A on the capacitor C_1 once it is charged up.

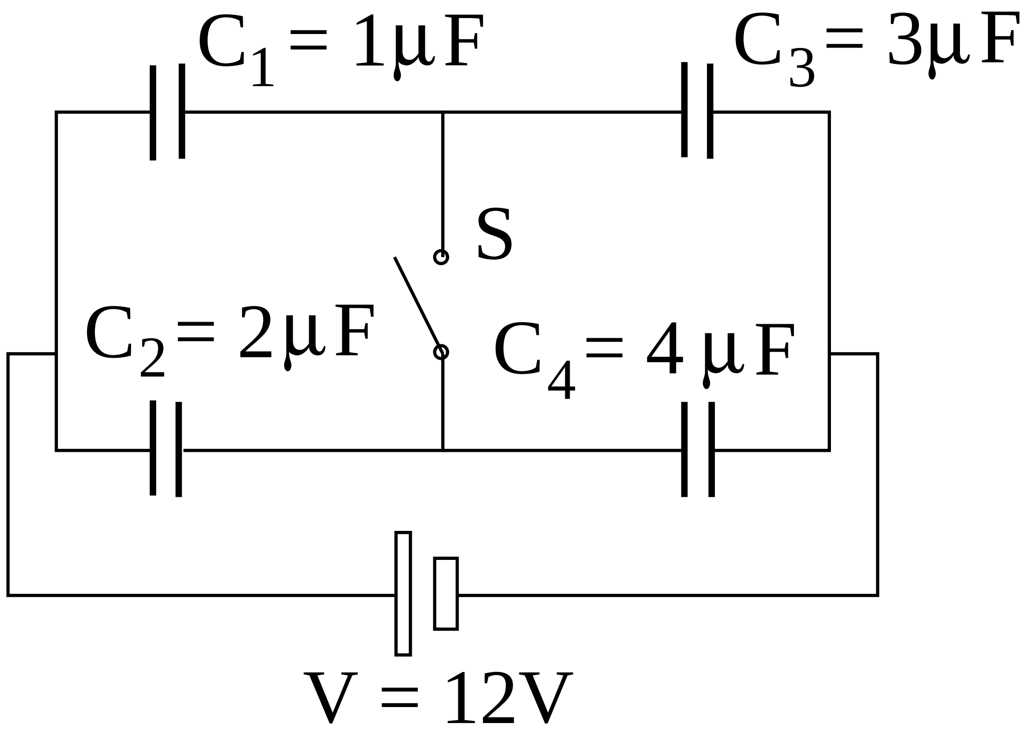
(b) Then the switch is thrown to B, which charges up the capacitors C_2 and C_3. The capacitor C_1 is partially discharged in the process. Find the charges Q_1, Q_2, Q_3 on all three capacitors and the voltages V_1, V_2, V_3 across each capacitor once equilibrium has been reached again. What is the energy U_B now stored in the circuit?



In the circuit shown find the charges Q_1, Q_2, Q_3, Q_4 on each capacitor and the voltages V_1, V_2, V_3, V_4 across each capacitor

(a) when the switch S is open,

(b) when the switch S is closed.



Capacitor Circuit (9)

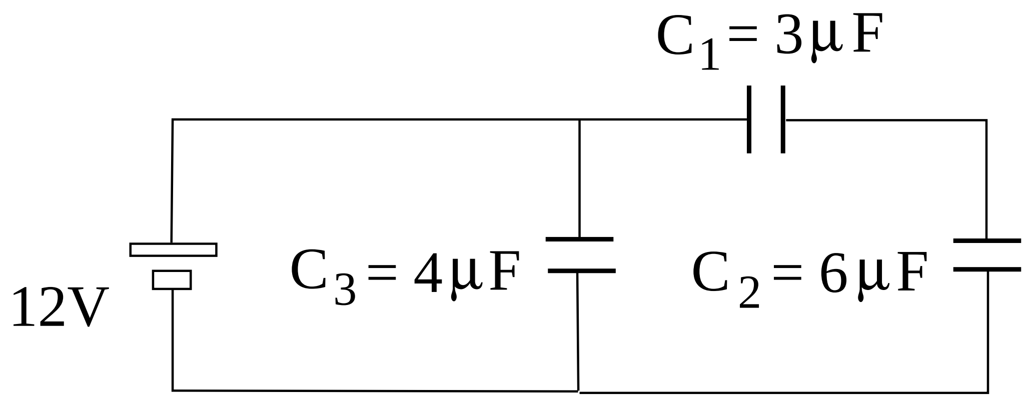
The circuit of capacitors connected to a battery is at equilibrium.

(a) Find the equivalent capacitance C_{eq}.

(b) Find the total energy U stored in the circuit (excluding the battery).

(c) Find the the charge Q_3 on capacitor C_3.

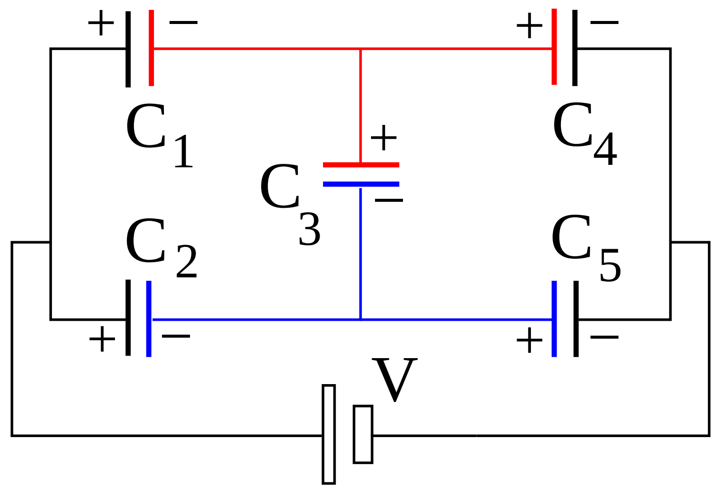
(d) Find the voltage V_2 across capacitor C_2.



More Complex Capacitor Circuit

No two capacitors are in parallel or in series.
Solution requires different strategy:

- zero charge on each conductor (here color coded),
- zero voltage around any closed loop.



Specifications: $C_1, \ldots, Q_5, V$.
Five equations for unknowns $Q_1, \ldots, Q_5$:

- $Q_1 + Q_2 - Q_4 - Q_5 = 0$
- $Q_3 + Q_4 - Q_1 = 0$
- $\dfrac{Q_5}{C_5} + \dfrac{Q_3}{C_3} - \dfrac{Q_4}{C_4} = 0$
- $\dfrac{Q_2}{C_2} - \dfrac{Q_1}{C_1} - \dfrac{Q_3}{C_3} = 0$
- $V - \dfrac{Q_1}{C_1} - \dfrac{Q_4}{C_4} = 0$

Equivalent capacitance: $C_{eq} = \dfrac{Q_1 + Q_2}{V}$

(a) $C_m = 1\text{pF}$, $m = 1, \ldots, 5$ and $V = 1\text{V}$:

$$C_{eq} = 1\text{pF}, \;\; Q_3 = 0,$$

$$Q_1 = Q_2 = Q_4 = Q_5 = \frac{1}{2}\text{pC}.$$

(b) $C_m = m\,\text{pF}$, $m = 1, \ldots, 5$ and $V = 1\text{V}$:

$$C_{eq} = \frac{159}{71}\text{pF}, \;\; Q_1 = \frac{55}{71}\text{pC}, \;\; Q_2 = \frac{104}{71}\text{pC},$$

$$Q_3 = -\frac{9}{71}\text{pC}, \;\; Q_4 = \frac{64}{71}\text{pC}, \;\; Q_5 = \frac{95}{71}\text{pC}.$$