Electromagnetic Plane Wave (1)

Maxwell's equations for electric and magnetic fields in free space (no sources):

- Gauss' laws: $\oint \vec{E} \cdot d\vec{A} = 0, \qquad \oint \vec{B} \cdot d\vec{A} = 0.$
- Faraday's and Ampère's laws: $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}, \qquad \oint \vec{B} \cdot d\vec{\ell} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt}.$

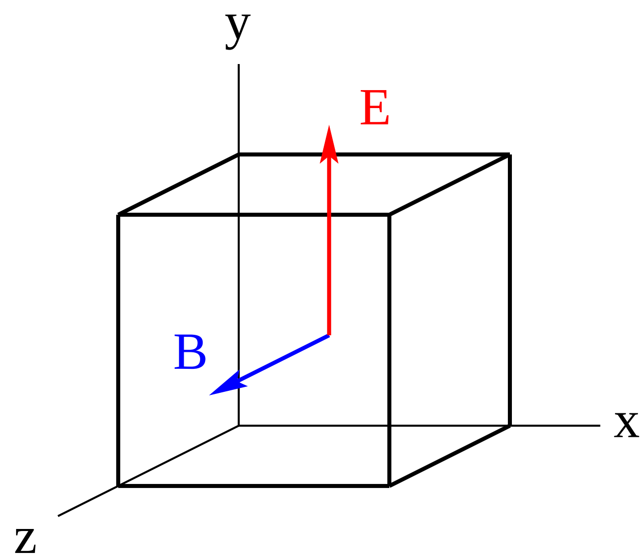
Consider fields of particular directions and dependence on space:

$$\vec{E} = E_y(x,t)\hat{j}, \quad \vec{B} = B_z(x,t)\hat{k}.$$

Gauss' laws are then automatically satisfied.

Use the cubic Gaussian surface to show that

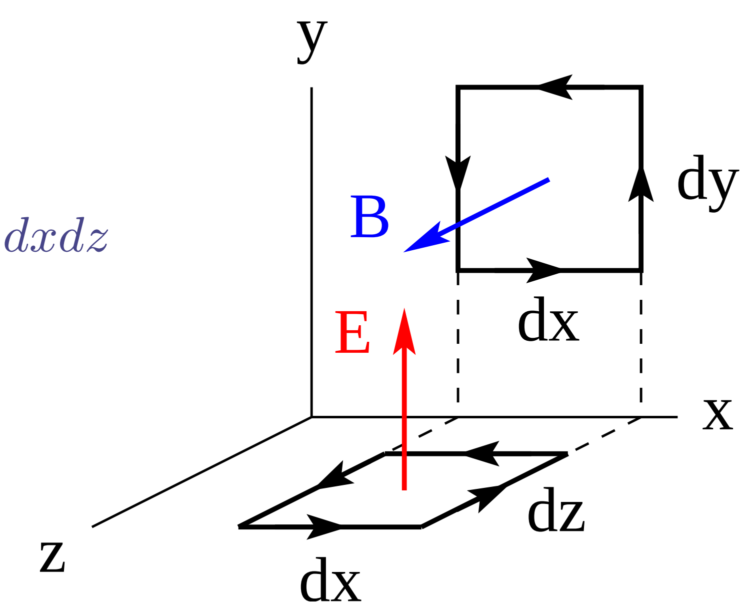
- the net electric flux Φ_E is zero,
- the net magnetic flux Φ_B is zero.



Electromagnetic Plane Wave (2)

- Faraday's law, $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$,
  applied to loop in (x, y)-plane becomes
  $$[E_y(x+dx,t) - E_y(x,t)]dy = -\frac{\partial}{\partial t} B_z(x,t)dxdy$$
  $$\Rightarrow \quad \frac{\partial}{\partial x} E_y(x,t) = -\frac{\partial}{\partial t} B_z(x,t) \qquad \text{(F)}$$

- Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$,
  applied to loop in (x, z)-plane becomes
  $$[-B_z(x+dx,t) + B_z(x,t)]dz = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_y(x,t)dxdz$$
  $$\Rightarrow \quad -\frac{\partial}{\partial x} B_z(x,t) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_y(x,t) \qquad \text{(A)}$$



Electromagnetic Plane Wave (3)

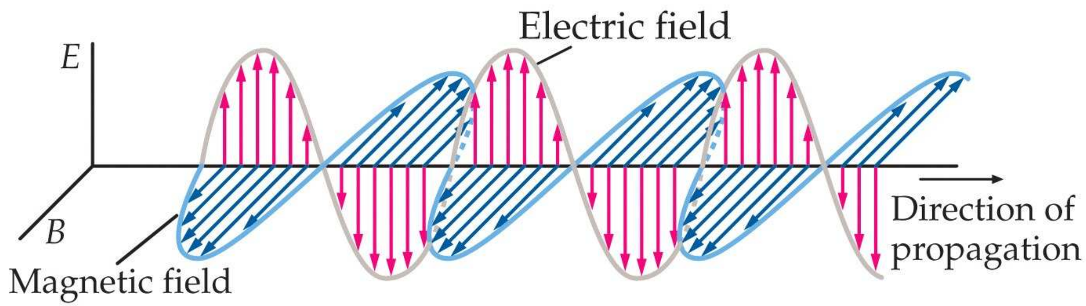
Take partial derivatives $\frac{\partial}{\partial x}$(F) and $\frac{\partial}{\partial t}$(A): $\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial t \partial x}, \quad -\frac{\partial^2 B_z}{\partial t \partial x} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$.

$\Rightarrow \frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2}$ (E) (wave equation for electric field).

Take partial derivatives $\frac{\partial}{\partial t}$(F) and $\frac{\partial}{\partial x}$(A): $\frac{\partial^2 E_y}{\partial t \partial x} = -\frac{\partial^2 B_z}{\partial t^2}, \quad -\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t \partial x}$.

$\Rightarrow \frac{\partial^2 B_z}{\partial t^2} = c^2 \frac{\partial^2 B_z}{\partial x^2}$ (B) (wave equation for magnetic field).

$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ (speed of light).



Sinusoidal solution:

- $E_y(x,t) = E_{max} \sin(kx - \omega t)$
- $B_z(x,t) = B_{max} \sin(kx - \omega t)$

Electromagnetic Plane Wave (4)

For given wave number k the angular frequency ω is determined, for example by substitution of $E_{max}\sin(kx-\omega t)$ into (E).

For given amplitude E_{max} the amplitude B_{max} is determined, for example, by substituting $E_{max}\sin(kx-\omega t)$ and $B_{max}\sin(kx-\omega t)$ into (A) or (F).

$$\Rightarrow \quad \frac{\omega}{k} = \frac{E_{max}}{B_{max}} = c.$$

The direction of wave propagation is determind by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\,\vec{E}\times\vec{B}.$$