Classical inverse scattering [mln104]

Goal: reconstruction of potential $V(r)$ from cross section $\sigma(\theta)$.

Assumptions: $dV/dr < 0$ (repulsive force), $V(0) > E$, $V(\infty) = 0$.

Consequence: $\theta(s_1) > \theta(s_2)$ if $s_1 < s_2$.

Calculate $s(\theta)$ from $\sigma(\theta)$: $2\pi \int_\theta^\pi d\theta \sin\theta\, \sigma(\theta) = 2\pi \int_0^s ds' s' = \pi s^2$.

Orbital integral from [mln20] with $u \doteq 1/r$:

$$\frac{\pi - \theta(s)}{2} = \int_0^{u_m} \frac{s du}{\sqrt{1 - \frac{V(1/u)}{E} - s^2 u^2}}, \qquad s^2 u_m^2 + V(1/u_m)/E = 1.$$

Transformation with $x \doteq 1/s^2$, $\tilde{\theta}(x) = \theta(s)$, and $w(u) \doteq \sqrt{1 - V(1/u)/E}$:

$$\frac{\pi - \tilde{\theta}(x)}{2} = \int_0^{u_m} \frac{du}{\sqrt{x[w(u)]^2 - u^2}}, \qquad u_m^2 = x[w(u_m)]^2 \quad \Rightarrow \; u_m(x).$$

Transformation: $\displaystyle \frac{1}{2}\int_0^\alpha dx \frac{\pi - \tilde{\theta}(x)}{\sqrt{\alpha - x}} = \int_0^\alpha dx \int_0^{u_m} \frac{du}{\sqrt{\big(x[w(u)]^2 - u^2\big)\big(\alpha - x\big)}}.$

$$\Rightarrow \; \pi\sqrt{\alpha} - \int_0^\alpha dx\, \tilde{\theta}'(x)\sqrt{\alpha - x} = \pi \int_0^{u_m(\alpha)} \frac{du}{w(u)}. \qquad \text{[mex243]}$$

Set $\alpha = u^2/w^2$, implying $u_m(\alpha) \to u$, take d/du, and multiply by du:

$$\frac{\pi}{w} dw = -d\left(\frac{u}{w}\right) \int_0^{u^2/w^2} dx \frac{\tilde{\theta}'(x)}{\sqrt{u^2/w^2 - x}}. \qquad \text{[mex244]}$$

Integrate differentials dw and $d(u/w)$ with consistent boundary values:

$$\Rightarrow \; w(u) = \exp\left(\frac{1}{\pi} \int_{w/u}^\infty ds \frac{\theta(s)}{\sqrt{s^2 - [w(u)]^2/u^2}} \right). \qquad \text{[mex245]}$$

The solution $w(u)$ of this integral equation for given $\theta(s)$ thus determines $V(r)$ from $\sigma(\theta)$.

[Landau and Lifshitz 1976]