Solid Sphere Rolling on Plane [mln106]

A solid sphere of mass m and radius a is rolling without slipping on the xy-plane under the influence of an external force $\mathbf{F} = (F_x, F_y, F_z)$ and an external torque $\mathbf{N} = (N_x, N_y, N_z)$, both acting on its center of mass.

The rolling motion is described by the instantaneous velocity $\mathbf{V} = (V_x, V_y, V_z)$ of the center of mass and the instantaneous angular velocity $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ about its center of mass.

Establishing the explicit dependences of $d\mathbf{V}/dt$ and $d\vec{\omega}/dt$ on $\mathbf{F}$ and $\mathbf{N}$ reduces the solution to quadrature.

Nonholonomic constraint: point of contact with plane is at rest: $\mathbf{V}+\vec{\omega}\times\mathbf{r} = 0$.
Unit vector $\hat{\mathbf{n}}$ directed from point of contact to center of mass: $\mathbf{r} = -a\hat{\mathbf{n}}$.

$$\Rightarrow \;\; \dot{\mathbf{V}} = a\,\dot{\vec{\omega}} \times \hat{\mathbf{n}}. \tag{1}$$

Equations of motion involve contact force $\mathbf{F}^{\mathrm{c}}$ and torque $\mathbf{r} \times \mathbf{F}^{\mathrm{c}}$:

- $d\mathbf{p}/dt = \mathbf{F}_{\mathrm{tot}}; \quad \mathbf{p} = m\mathbf{V}, \quad \mathbf{F}_{\mathrm{tot}} = \mathbf{F} + \mathbf{F}^{\mathrm{c}},$
- $d\mathbf{L}/dt = \mathbf{N}_{\mathrm{tot}}; \quad \mathbf{L} = I\vec{\omega}, \quad \mathbf{N}_{\mathrm{tot}} = \mathbf{N} + \mathbf{r} \times \mathbf{F}^{\mathrm{c}},$

where $I = \frac{2}{5}ma^2$ is the moment of inertia for a solid sphere.

$$\Rightarrow \;\; m\frac{d\mathbf{V}}{dt} = \mathbf{F} + \mathbf{F}^{\mathrm{c}}, \quad I\frac{d\vec{\omega}}{dt} = \mathbf{N} - a\hat{\mathbf{n}} \times \mathbf{F}^{\mathrm{c}}. \tag{2}$$

Eliminate $\mathbf{F}^{\mathrm{c}}$ in (2) using (1) [mex260]:

$$\dot{V}_x = \frac{5}{7ma}(aF_x + N_y), \quad \dot{V}_y = \frac{5}{7ma}(aF_y - N_x), \quad \dot{V}_z = 0,$$

$$\dot{\omega}_x = -\frac{5}{7ma^2}(aF_x + N_y), \quad \dot{\omega}_y = \frac{5}{7ma^2}(aF_y - N_x), \quad \dot{\omega}_z = \frac{5}{2ma^2}N_z.$$