Noether's Theorem I [mln12]

Symmetries indicated by cyclic variables in the Lagrangian lead to conservation laws. Some symmetries may be obscure or hidden. Noether's theorem derives conservation laws from a general class of continuous symmetries.

Consider a Lagrangian system $L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t)$.

Theorem (restricted case):

If a transformation $Q_i(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t,\epsilon),\ i=1,\ldots,n$ with $Q_i=q_i$ at $\epsilon=0$ can be found such that

$$\left.\frac{\partial L'}{\partial \epsilon}\right|_{\epsilon=0}=0$$

is satisfied, where

$$L'(Q_1,\ldots,Q_n,\dot{Q}_1,\ldots,\dot{Q}_n,t,\epsilon)\doteq L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t),$$

then the following quantity is conserved:

$$I(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t)=\sum_i \frac{\partial L}{\partial \dot{q}_i}\left.\frac{\partial q_i}{\partial \epsilon}\right|_{\epsilon=0}.$$

Proof:

Use the inverse transformation $q_i(Q_1,\ldots,Q_n,\dot{Q}_1,\ldots,\dot{Q}_n,t,\epsilon),\ i=1,\ldots,n$ and keep the variables $Q_i,\dot{Q}_i$ fixed.

$$\frac{\partial L'}{\partial \epsilon}=\sum_i\left[\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon}+\frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \epsilon}\right]=\sum_i\left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \epsilon}+\frac{\partial L}{\partial \dot{q}_i}\left(\frac{d}{dt}\frac{\partial q_i}{\partial \epsilon}\right)\right]=0.$$

$$\Rightarrow\ \frac{\partial L'}{\partial \epsilon}=\frac{d}{dt}\left[\sum_i\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \epsilon}\right]=0.$$

Applications:

- translation in space [mex35]
- rotation in space [mex36]