Noether’s Theorem I

Symmetries indicated by cyclic variables in the Lagrangian lead to conservation laws. Some symmetries may be obscure or hidden. Noether’s theorem derives conservation laws from a general class of continuous symmetries.

Consider a Lagrangian system $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

**Theorem** (restricted case):

If a transformation $Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon)$, $i = 1, \ldots, n$ with $Q_i = q_i$ at $\epsilon = 0$ can be found such that

$$\frac{\partial L'}{\partial \epsilon} \bigg|_{\epsilon=0} = 0$$

is satisfied, where

$$L'(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon) \doteq L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t),$$

then the following quantity is conserved:

$$I(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \epsilon} \bigg|_{\epsilon=0}.$$

**Proof:**

Use the inverse transformation $q_i(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon)$, $i = 1, \ldots, n$ and keep the variables $Q_i, \dot{Q}_i$ fixed.

$$\frac{\partial L'}{\partial \epsilon} = \sum_i \left[ \frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \epsilon} \right] = \sum_i \left[ \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial q_i} \left( \frac{d}{dt} \frac{\partial q_i}{\partial \epsilon} \right) \right] = 0.

\Rightarrow \frac{\partial L'}{\partial \epsilon} = \frac{d}{dt} \left[ \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \epsilon} \right] = 0.$$

**Applications:**

- translation in space [mex35]
- rotation in space [mex36]