Variational Problem with Auxiliary Condition

Search for a function $y(x)$ that yields an extremum of the integral

$$J = \int_{x_1}^{x_2} dx \ f[y(x), y'(x); x]$$

subject to an auxiliary condition in the form of the integral constraint

$$C = \int_{x_1}^{x_2} dx \ \sigma[y(x), y'(x); x] = \text{const.}$$

Use the functional $F_\lambda[y(x), y'(x); x] = f[y(x), y'(x); x] + \lambda \sigma[y(x), y'(x); x]$, where $\lambda$ is an undetermined Lagrange multiplier.

Find the extremum of $J_\lambda = \int_{x_1}^{x_2} dx \ F_\lambda[y(x), y'(x); x]$.

This leads to Euler’s equation

$$\frac{\partial F_\lambda}{\partial y} - \frac{d}{dx} \left( \frac{\partial F_\lambda}{\partial y'} \right) = 0.$$ 

Then adjust the value of $\lambda$ in the solution such that the auxiliary condition is satisfied.

Examples:

- Isoperimetric problem [mex28]
- Catenary problem [mex38]