Generalized Forces of Constraint and Hamilton's Principle [mln17]

Lagrangian: $L(q_1, q_2, \dot{q}_1, \dot{q}_2, t)$.

Holonomic constraint: $f(q_1, q_2, t) = 0$.

Action integral: $J(\alpha) = \int_{t_1}^{t_2} dt\, L(q_1, q_2, \dot{q}_1, \dot{q}_2, t), \quad q_i(t,\alpha) = q_i(t,0) + \alpha\eta_i(t)$.

$$\Rightarrow \frac{dJ}{d\alpha} = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} \right) \frac{\partial q_1}{\partial \alpha} + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} \right) \frac{\partial q_2}{\partial \alpha} \right]_{\alpha=0} = 0.$$

Constraint: $\displaystyle \frac{df}{d\alpha} = \frac{\partial f}{\partial q_1}\frac{\partial q_1}{\partial \alpha} + \frac{\partial f}{\partial q_2}\frac{\partial q_2}{\partial \alpha} = 0 \quad \Rightarrow \eta_2(t) = -\eta_1(t)\frac{\partial f/\partial q_1}{\partial f/\partial q_2}$.

$$\Rightarrow \frac{dJ}{d\alpha} = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} \right) - \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} \right) \frac{\partial f/\partial q_1}{\partial f/\partial q_2} \right] \eta_1(t) = 0.$$

$$\Rightarrow \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} \right) \left(\frac{\partial f}{\partial q_1} \right)^{-1} = \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} \right) \left(\frac{\partial f}{\partial q_2} \right)^{-1} = -\lambda(t).$$

This results in 3 equations for the unknown functions $q_1(t), q_2(t), \lambda(t)$:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \lambda(t)\frac{\partial f}{\partial q_i} = 0, \ i = 1, 2; \quad f(q_1, q_2, t) = 0.$$

Generalized forces of constraint: $\displaystyle Q_i(t) = \lambda(t)\frac{\partial f}{\partial q_i}, \ i = 1, 2.$

Generalization to n coordinates and k constraints:

$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ with $f_j(q_1, \ldots, q_n, t) = 0, \ j = 1, \ldots, k$.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_j \lambda_j(t)\frac{\partial f_j}{\partial q_i} = 0, \ i = 1, \ldots, n,$$

$$\sum_i \frac{\partial f_j}{\partial q_i} dq_i + \frac{\partial f_j}{\partial t} dt = 0, \ j = 1, \ldots, k.$$

Applications:

- Static frictional force of constraint [mex32].
- Normal force of constraint [mex33]
- Particle sliding down sphere [mex34]
- Particle sliding inside cone: normal force of constraint [mex159]