Central Force Problem: Formal Solution [mln18]

Lagrangian: $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) - V(r)$.

Lagrange equations (coupled 2^{nd} order ODEs):

$$m\ddot{r} = mr\dot{\vartheta}^2 - \frac{\partial V}{\partial r}, \qquad \frac{d}{dt}\left(mr^2\dot{\vartheta}\right) = 0.$$

Integrals of the motion (angular momentum and energy):

$$[A]\ \ \ell = mr^2\dot{\vartheta} = \text{const}, \quad [B]\ \ E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) = \text{const}.$$

Motion in time (solution by quadrature):

$$[B]\ \ \frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]} \qquad \Rightarrow\ t = \pm\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}}$$

$$\Rightarrow\ r(t) = \ldots$$

$$[A]\ \ \frac{d\vartheta}{dt} = \frac{\ell}{mr^2} \quad \Rightarrow\ \vartheta(t) = \frac{\ell}{m}\int_0^t \frac{dt}{r^2(t)} + \vartheta_0.$$

Integration constants: $E,\ \ell,\ r_0,\ \vartheta_0$.

Orbital integral: eliminate t from $r(t), \vartheta(t)$ to obtain $r(\vartheta)$ or $\vartheta(r)$.

$$\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]} = \frac{dr}{dt} = \frac{dr}{d\vartheta}\frac{d\vartheta}{dt} = \frac{dr}{d\vartheta}\frac{\ell}{mr^2}.$$

$$\Rightarrow\ \int_{r_0}^{r} dr \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} = \int_{\vartheta_0}^{\vartheta} d\vartheta = \vartheta - \vartheta_0 \quad \Rightarrow\ \vartheta(r) = \vartheta_0 + \ldots$$

Orbital integral for power-law potentials $V(r) = -\frac{\kappa}{r^\alpha}$: set $u \doteq 1/r$.

$$\vartheta - \vartheta_0 = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2m\kappa}{\ell^2}\,u^\alpha - u^2}}.$$

For the cases $\alpha = 6, 4, 3, 2, 1, -1, -2, -4, -6$, the orbit can be expressed in terms of elementary functions.