Motion in rotating frame of reference [mln22]

Consider two frames of reference with identical origins. Frame R is rotating with angular velocity $\vec{\omega}$ relative to the inertial frame I.

Coordinate axes: $\mathbf{e}_i^{(I)} = \text{const}, \quad \dot{\mathbf{e}}_i^{(R)} = \vec{\omega} \times \mathbf{e}_i^{(R)}, \quad i = 1, 2, 3.$

Kinematics of a particle moving in frame R.

- Position: $\mathbf{r}_R = \mathbf{r}_I \doteq \mathbf{r}, \quad$ where $\mathbf{r}_I = \sum_{i=1}^{3} x_i^{(I)} \mathbf{e}_i^{(I)}, \quad \mathbf{r}_R = \sum_{i=1}^{3} x_i^{(R)} \mathbf{e}_i^{(R)}.$

- Velocity: $\dfrac{d\mathbf{r}}{dt} = \sum_{i=1}^{3} \dot{x}_i^{(I)} \mathbf{e}_i^{(I)} = \sum_{i=1}^{3} \left[\dot{x}_i^{(R)} \mathbf{e}_i^{(R)} + x_i^{(R)} \dot{\mathbf{e}}_i^{(R)} \right].$

$$\Rightarrow \sum_{i=1}^{3} \dot{x}_i^{(I)} \mathbf{e}_i^{(I)} = \sum_{i=1}^{3} \left[\dot{x}_i^{(R)} \mathbf{e}_i^{(R)} + \vec{\omega} \times x_i^{(R)} \mathbf{e}_i^{(R)} \right] \qquad \Rightarrow \mathbf{v}_I = \mathbf{v}_R + \vec{\omega} \times \mathbf{r}.$$

- Acceleration: $\dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d\mathbf{v}_I}{dt} = \left(\dfrac{d\mathbf{v}_R}{dt} \right)_I + \dot{\vec{\omega}} \times \mathbf{r} + \vec{\omega} \times \left(\dfrac{d\mathbf{r}}{dt} \right)_I.$

with $\left(\dfrac{d\mathbf{v}_R}{dt} \right)_I = \mathbf{a}_R + \vec{\omega} \times \mathbf{v}_R, \quad \vec{\omega} \times \left(\dfrac{d\mathbf{r}}{dt} \right)_I = \vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$

$$\Rightarrow \mathbf{a}_I = \mathbf{a}_R + \dot{\vec{\omega}} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$$

Dynamics of a particle of mass m.

- Inertial frame: $m\mathbf{a}_I = \mathbf{F}_I$
- Rotating frame: $m\mathbf{a}_R = \mathbf{F}_I - m\dot{\vec{\omega}} \times \mathbf{r} - 2m\vec{\omega} \times \mathbf{v}_R - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$

Real and fictitious forces:

- $\mathbf{F}_I$ (applied force).
- $-m\dot{\vec{\omega}} \times \mathbf{r}$ (due to angular acceleration of frame R).
- $-2m\vec{\omega} \times \mathbf{v}_R$ (Coriolis force).
- $-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$ (centrifugal force).

If the origin of frame R undergoes a lateral motion in addition to the rotation, then a term $-m(d^2\mathbf{R}/dt^2)$ must be added to the fictitious forces. Here $\mathbf{R}$ is the vector pointing from the origin of frame I to the origin of frame R.