Holonomic constraints in rotating frame [mln23]

Recipe for solving a Lagrangian mechanics problem with holonomic constraints between coordinates in the rotating frame.

- Formulate Lagrangian in inertial frame (I) without imposing constraints.
- Transform coordinates to the rotating frame (R).
- Impose holonomic constraints via independent generalized coordinates.
- Derive Lagrange equations in frame R.

Example: particle of mass m moving on surface of rotating Earth in vertical plane parallel to meridian and subject to scalar potential V.

- Lagrangian: $L_I = \dfrac{1}{2}m(\dot{x_I}^2 + \dot{y_I}^2 + \dot{z_I}^2) - V(x_I, y_I, z_I)$.
- Earth's angular velocity: $\vec{\omega} = (-\omega\cos\lambda, 0, \omega\sin\lambda)$.
- Transformation: $\mathbf{v}_I = \mathbf{v} + \vec{\omega}\times\mathbf{r} = \begin{pmatrix} \dot{x} - \omega y\sin\lambda \\ \dot{y} + \omega x\sin\lambda + \omega z\cos\lambda \\ \dot{z} - \omega y\cos\lambda \end{pmatrix}$.
- Constraint: $y = 0 \quad \Rightarrow \quad \mathbf{v}_I = (\dot{x}_I, \dot{y}_I, \dot{z}_I) = \begin{pmatrix} \dot{x} \\ \omega x\sin\lambda + \omega z\cos\lambda \\ \dot{z} \end{pmatrix}$.
- Substitute $\mathbf{v}_I$ into Lagrangian: $L_I = L(x, z, \dot{x}, \dot{z})$.
- Lagrange equations: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x} = 0, \quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{z}} - \dfrac{\partial L}{\partial z} = 0$.

Notes:

- The accelerated translational motion can be taken into account by a modified acceleration due to gravity: $\mathbf{g} = \mathbf{g}_0 + \omega^2\mathbf{r}_\perp$ [mex170].
- In the local coordinate system, $\mathbf{e}_x$ is pointing south, $\mathbf{e}_y$ is pointing east, and $\mathbf{e}_z$ is pointing vertically up.
- It is common practice to drop subscripts R in the rotating frame to keep the notation simple.