Rotational Kinetic Energy [mln25]

Consider a rigid body undergoing a purely rotational motion ($\dot{\mathbf{R}} = 0$).

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left(\vec{\omega} \times \mathbf{r}_\alpha\right)^2 = \frac{1}{2}\sum_\alpha m_\alpha \left[\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\mathbf{r}_\alpha)^2\right].$$

$\vec{\omega} = (\omega_1, \omega_2, \omega_3)$: instantantaneous angular velocity of body frame relative to inertial frame (components in body frame).
$\mathbf{r}_\alpha = (r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3})$: position coordinates in body frame.

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left[\left(\sum_i \omega_i^2\right)\left(\sum_k r_{\alpha k}^2\right) - \left(\sum_i \omega_i r_{\alpha i}\right)\left(\sum_j \omega_j r_{\alpha j}\right)\right].$$

Use $\omega_i = \sum_j \omega_j \delta_{ij}$.

$$T = \frac{1}{2}\sum_\alpha \sum_{ij} m_\alpha \left[\omega_i\omega_j\delta_{ij} r_\alpha^2 - \omega_i\omega_j r_{\alpha i} r_{\alpha j}\right] = \frac{1}{2}\sum_{ij} I_{ij}\,\omega_i\omega_j.$$

Inertia tensor: $I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} r_\alpha^2 - r_{\alpha i} r_{\alpha j}\right]$.

Inertia tensor of cont. mass distrib. $\rho(\mathbf{r})$: $I_{ij} = \int d^3r\, \rho(\mathbf{r}) \left[\delta_{ij} r^2 - r_i r_j\right]$.

Comments:

- Inertia tensor is symmetric: $I_{ij} = I_{ji}$.
- Matrix notation: $T = \frac{1}{2}(\omega_1, \omega_2, \omega_3) \cdot \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$.
- I_{ij} depends on choice of body frame.
- If $\omega = \omega_i$ then $T = \frac{1}{2} I_{ii}\omega_i^2$, where I_{ii} is called a moment of inertia.
- The use of the body frame guarantees that the inertia tensor is time-independent.

For $\dot{\mathbf{R}} \neq 0$ the kinetic energy generally has three parts, *translational, mixed*, and *rotational*, which are further discussed in [mex67], [mex173]:

$$T = \sum_\alpha \frac{1}{2} m_\alpha \left(\dot{\mathbf{R}} + \vec{\omega}\times\mathbf{r}_\alpha\right)^2 = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{R}}^2 + \sum_\alpha m_\alpha \dot{\mathbf{R}}\cdot\vec{\omega}\times\mathbf{r}_\alpha + \frac{1}{2}\sum_\alpha m_\alpha \left(\vec{\omega}\times\mathbf{r}_\alpha\right)^2.$$