Euler's Equations [mln27]

Equation of motion in inertial frame: $\left(\frac{d\mathbf{L}}{dt}\right)_I = \mathbf{N}.$

Equation of motion in (rotating) body frame: $\left(\frac{d\mathbf{L}}{dt}\right)_R + \vec{\omega} \times \mathbf{L} = \mathbf{N}.$

Proof:

$$\mathbf{N} = \frac{d}{dt}\left[\sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha)\right] = \frac{d}{dt}\left[\sum_i L_i \mathbf{e}_i\right] = \frac{d}{dt}\left[\sum_{ij} I_{ij}\omega_j \mathbf{e}_i\right].$$

Use $\dot{\mathbf{e}}_i = \vec{\omega} \times \mathbf{e}_i$. $\Rightarrow$ $\mathbf{N} = \sum_{ij} I_{ij}\dot{\omega}_j \mathbf{e}_i + \vec{\omega} \times \sum_{ij} I_{ij}\omega_j \mathbf{e}_i.$

Choose body frame with principal coordinate axes: $L_i = I_i\omega_i, i = 1, 2, 3.$

Euler's equations:

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= N_1 \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= N_2 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= N_3 \end{aligned}$$

A purely rotating rigid body has 3 degrees of freedom.
The associated Lagrange equations are three 2^{nd} order ODEs.

The solution via Euler's equations proceeds in two steps:

1. Euler's equations themselves are three 1^{st} order ODEs for $\omega_1, \omega_2, \omega_3$.
2. The transformation to the inertial frame, $\omega_i = \omega_i(\phi, \theta, \psi; \dot{\phi}, \dot{\theta}, \dot{\psi})$ amounts to solving another three 1^{st} order ODEs.