Driven Harmonic Oscillator II [mln29]

Equation of motion: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A(t)$.

Parameters: $\beta \doteq \gamma/2m,\ \omega_0 \doteq \sqrt{k/m},\ A(t) \doteq F(t)/m$.

Periodic driving force: $F(t+\tau) = F(t)$.

Fourier series: $A(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega t + b_n \sin n\omega t\right], \quad \omega = \frac{2\pi}{\tau}$.

Fourier coefficients: $a_n = \frac{2}{\tau}\int_0^\tau dt\, A(t)\cos n\omega t, \quad b_n = \frac{2}{\tau}\int_0^\tau dt\, A(t)\sin n\omega t$.

Linear response of system to periodic driving force:

$$x(t) = \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty} d(\omega_n)\left[a_n\cos(\omega_n t + \delta_n) + b_n \sin(\omega_n t + \delta_n)\right],$$

$$\omega_n = \frac{2\pi n}{\tau}, \quad d(\omega_n) = \frac{1}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}}, \quad \delta_n = \arctan\left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right).$$

Aperiodic driving force: $F(t) = mA(t)$ with $\int_{-\infty}^{+\infty} dt\, |A(t)| < \infty$.

Fourier transform: $\tilde{x}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}x(t), \quad \tilde{A}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}A(t)$.

Inverse transform: $x(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{x}(\omega), \quad A(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{A}(\omega)$.

Fourier transformed equation of motion is algebraic (not differential):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A(t) \ \Rightarrow\ -\omega^2\tilde{x}(\omega) - 2i\beta\omega\tilde{x}(\omega) + \omega_0^2\tilde{x}(\omega) = \tilde{A}(\omega).$$

Linear response of system to aperiodic driving force:

$$\tilde{x}(\omega) = \frac{\tilde{A}(\omega)}{\omega_0^2 - \omega^2 - 2i\beta\omega} \quad \Rightarrow\ x(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{x}(\omega).$$