Transformation to Principal Axes [mln30]

Solving the equations $\displaystyle\sum_{j=1}^{n}(k_{ij}-\omega_r^2 m_{ij})A_{jr}=0, \quad i,r=1,\ldots,n$
for the amplitudes A_{jr} of the n normal modes amounts to finding an *orthogonal* matrix $\mathbf{A}$, which diagonalizes the *symmetric* matrices $\mathbf{m}$ and $\mathbf{k}$ simultaneously:

$$\mathbf{A}^T\cdot\mathbf{m}\cdot\mathbf{A}=\mathbf{1}\doteq\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix},$$

$$\mathbf{A}^T\cdot\mathbf{k}\cdot\mathbf{A}=\mathbf{\Omega}^2\doteq\begin{pmatrix}\omega_1^2 & 0 & \cdots & 0\\ 0 & \omega_2^2 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & \omega_n^2\end{pmatrix},$$

Normal coordinates: $\displaystyle Q_j=\sum_{i=1}^{n}A_{ij}q_i$.

Lagrangian: $\displaystyle L=\frac{1}{2}\sum_{ij}\left(m_{ij}\dot{q}_i\dot{q}_j-k_{ij}q_iq_j\right)=\frac{1}{2}\sum_{r=1}^{n}\left(\dot{Q}_r^2-\omega_r^2Q_r^2\right).$

Lagrange equations: $\displaystyle\sum_{j=1}^{n}\left(m_{ij}\ddot{q}_j+k_{ij}q_j\right)=0, \quad i=1,\ldots,n$ (coupled).

Lagrange equations: $\ddot{Q}_r+\omega_r^2Q_r=0, \quad r=1,\ldots,n$ (decoupled).

Applications:

- Blocks and springs in series [mex123]
- Small oscillations of the double pendulum [mex124]
- Two coupled oscillators [mex186]