Differential Constraints [mln37]

A general class of constraints can be expressed in differential form:

$$\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, \quad j = 1, \ldots, m.$$

n: number of generalized coordinates (Cartesian, polar, or other).
m: number of independent differential constraints.
$n - m$: number of degrees of freedom.

Integrability condition of differential constraints:

$$\frac{\partial a_{ji}}{\partial q_k} = \frac{\partial a_{jk}}{\partial q_i}, \quad \frac{\partial a_{ji}}{\partial q_t} = \frac{\partial a_{jt}}{\partial q_i}, \quad j = 1, \ldots, m, \quad i, k = 1, \ldots, n.$$

Holonomic constraints: Integrability condition is satisfied.

$$a_{ji} = \frac{\partial f_j}{\partial q_i}, \quad a_{jt} = \frac{\partial f_j}{\partial t} \ \Rightarrow \ f_j(q_1, \ldots, q_n, t) = 0, \ j = 1, \ldots, m.$$

A set of m generalized coordinates q_i can be eliminated.

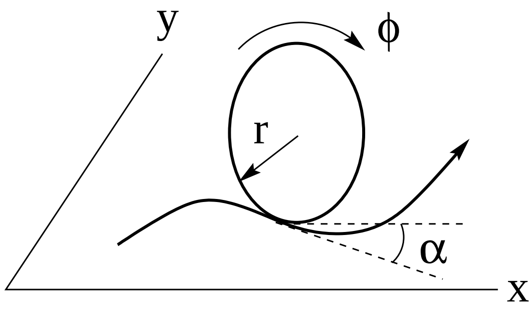
Nonholonomic constraints: Integrability condition is violated.

All n generalized coordinates are required. The kinematic effect of a nonholonomic constraint is to restrict the direction of the allowable motions at any given point in n-dimensional configuration space. This restriction does not reduce the dimensionality of the configuration space.

Example: Disk rolling upright without slipping on horizontal plane.

Generalized coordinates:
x, y (position),
ϕ, α (orientation).

Constraints:
$dx - r \cos\alpha \, d\phi = 0$,
$dy + r \sin\alpha \, d\phi = 0$.



None of these coordinates can be eliminated. It is possible to arrive at any configuration (x, y, α, ϕ) from any other configuration via a path that satisfies the two constraints.