Routhian Function

Goal: systematic elimination of cyclic coordinates in the Lagrangian formulation of mechanics.

Consider a system with \( n \) generalized coordinates of which the first \( k \) are cyclic.

Lagrangian: \( L(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \Rightarrow q_1, \ldots, q_k \) are cyclic.

Routhian: \( R(q_{k+1}, \ldots, q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, \beta_1, \ldots, \beta_k, t) = L - \sum_{i=1}^{k} \beta_i \dot{q}_i \)

where the relations \( \beta_i = \frac{\partial L}{\partial \dot{q}_i} = \text{const}, \; i = 1, \ldots, k \) are to be inverted into \( \dot{q}_i = \dot{q}_i(q_{k+1}, \ldots, q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, \beta_1, \ldots, \beta_k, t), \; i = 1, \ldots, k. \)

Compare coefficients of the variations

\[
\delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i} \delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta_i} \delta \beta_i + \frac{\partial R}{\partial t} \delta t,
\]

\[
\delta \left( L - \sum_{i=1}^{k} \beta_i \dot{q}_i \right) = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \sum_{i=1}^{k} \dot{q}_i \delta \beta_i + \frac{\partial L}{\partial t} \delta t.
\]

Resulting relations between partial derivatives:

\[
\frac{\partial R}{\partial q_i} = \frac{\partial L}{\partial q_i}, \quad \frac{\partial R}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}, \quad i = k + 1, \ldots, n,
\]

\[
\frac{\partial R}{\partial t} = \frac{\partial L}{\partial t}; \quad \dot{q}_i = -\frac{\partial R}{\partial \beta_i}, \quad i = 1, \ldots, k.
\]

Lagrange equations for the noncyclic coordinates:

\[
\frac{\partial R}{\partial q_i} - \frac{d}{dt} \frac{\partial R}{\partial \dot{q}_i} = 0, \quad i = k + 1, \ldots, n.
\]

Time evolution of cyclic coordinates:

\[
q_i(t) = -\int dt \frac{\partial R}{\partial \beta_i}, \quad i = 1, \ldots, k.
\]