Noether's Theorem III [mln42]

The continuous symmetry transformation may also involve the time.

Consider again a Lagrangian system $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

Theorem (most general case):

If a transformation $Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon)$, $T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon)$, $i = 1, \ldots, n$ with $Q_i = q_i$ and $T = t$ at $\epsilon = 0$ can be found such that

$$\frac{\partial}{\partial \epsilon} \left[L\left(Q_1, \ldots, Q_n, \frac{dQ_1}{dT}, \ldots, \frac{dQ_n}{dT}, T \right) \frac{dT}{dt} \right]_{\epsilon=0} = \frac{d}{dt}\, G(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$

is satisfied (for an arbitrary function G), then the following quantity is conserved:

$$I = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \left(\frac{\partial T}{\partial \epsilon} \right)_{\epsilon=0} - G + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \left[\left(\frac{\partial Q_i}{\partial \epsilon} \right)_{\epsilon=0} - \dot{q}_i \left(\frac{\partial T}{\partial \epsilon} \right)_{\epsilon=0} \right].$$

Proof:

$$\dot{G} = \left\{ \left[\sum_i \frac{\partial L}{\partial Q_i} \frac{\partial Q_i}{\partial \epsilon} + \sum_i \frac{\partial L}{\partial (dQ_i/dT)} \frac{\partial (dQ_i/dT)}{\partial \epsilon} + \frac{\partial L}{\partial T} \frac{\partial T}{\partial \epsilon} \right] \dot{T} + L \frac{\partial \dot{T}}{\partial \epsilon} \right\}_{\epsilon=0}.$$

Define: $A_i \equiv \left(\frac{\partial Q_i}{\partial \epsilon} \right)_{\epsilon=0}$, $\quad B \equiv \left(\frac{\partial T}{\partial \epsilon} \right)_{\epsilon=0}$.

Expand $Q_i = q_i + A_i \epsilon + \ldots$, $\quad T = t + B\epsilon + \ldots$.

$$\Rightarrow\ \dot{Q}_i = \dot{q}_i + \dot{A}_i \epsilon + \ldots, \quad \dot{T} = 1 + \dot{B}\epsilon + \ldots, \quad \frac{dQ_i}{dT} = \frac{\dot{Q}_i}{\dot{T}} = \frac{\dot{q}_i + \dot{A}_i \epsilon + \ldots}{\dot{t} + \dot{B}\epsilon + \ldots}.$$

At $\epsilon = 0$: $\dot{Q}_i = \dot{q}_i$, $\dot{T} = 1$, $\frac{dQ_i}{dT} = \dot{q}_i$, $\frac{\partial \dot{T}}{\partial \epsilon} = \dot{B}$, $\frac{\partial}{\partial \epsilon} \frac{dQ_i}{dT} = \dot{A}_i - \dot{q}_i \dot{B}$.

Use $\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$, $\quad \frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$.

$$\Rightarrow\ \frac{d}{dt} \left[LB - G + \sum_i \frac{\partial L}{\partial \dot{q}_i} (A_i - \dot{q}_i B) \right] = 0 \ \Rightarrow\ I = \text{const.}$$