Small Oscillations [mln43]

Consider undamped small-amplitude motion about a stable equilibrium.

Lagrangian: $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij} k_{ij} q_i q_j.$

Mass coefficients: $m_{ij} = \sum_{k=1}^{3N} m_k \left(\frac{\partial x_k}{\partial q_i}\right)_0 \left(\frac{\partial x_k}{\partial q_j}\right)_0.$

Stiffness coefficients: $k_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0.$

Lagrange equations are linear: $\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n} k_{ij} q_j = 0, \quad i = 1, \ldots, n.$

The matrices $\{m_{ij}\}$ and $\{k_{ij}\}$ are symmetric.

Ansatz for solution: $q_j(t) = A_j \cos(\omega t + \phi), \quad j = 1, \ldots, n.$

$\Rightarrow \sum_{j=1}^{n} (k_{ij} - \omega^2 m_{ij}) A_j \cos(\omega t + \phi) = 0, \quad i = 1, \ldots, n.$

Linear homogeneous equations:

$$\sum_{j=1}^{n} (k_{ij} - \omega^2 m_{ij}) A_j = 0, \quad i = 1, \ldots, n. \tag{1}$$

Characteristic equation (n^{th}-order polynomial in ω^2):

$$\begin{vmatrix} (k_{11} - \omega^2 m_{11}) & \cdots & (k_{1n} - \omega^2 m_{1n}) \\ \vdots & & \vdots \\ (k_{n1} - \omega^2 m_{n1}) & \cdots & (k_{nn} - \omega^2 m_{nn}) \end{vmatrix} = 0.$$

The n roots $\omega_1^2, \ldots, \omega_n^2$ of the characteristic equation are the eigenvalues associated with n *natural modes of vibration* (normal modes).

The normal mode with angular frequency ω_k is specified by a set of amplitudes $A_1^{(k)}, \ldots, A_n^{(k)}$. The amplitude ratios for this normal mode are determined from Eqs. (1):

$$\sum_{j=1}^{n} (k_{ij} - \omega_k^2 m_{ij}) A_j^{(k)} = 0, \quad i = 1, \ldots, n.$$