Bertrand's Theorem [mln44]

The only central force potentials $V(r)$ for which all bounded orbits are closed are the following:

- Kepler system: $V(r) = -\dfrac{\kappa}{r}$ (ellipses with $r = 0$ at one focus)
- Harmonic oscillator: $V(r) = \kappa' r^2$ (ellipses with $r = 0$ at center)

J. Bertrand's proof of 1873 is based on a 2^{nd} order perturbation calculation about stable circular orbits. The following derivation follows Arnold [1989] and rests on five lemmas:

1. The central force potential $V(r)$ has a circular orbit at $r = R$ if $V'(R) = \ell^2/mR^3$. This circular orbit is stable if $V''(R) + (3/R)V'(R) > 0$. [mex53] [mex125]
2. For a central force potential $V(r)$ with a circular orbit at $r = R$, the apsidal angle for orbits in the vicinity of this circular orbit is $\Delta\vartheta = \pi\sqrt{V'(R)/[3V'(R) + RV''(R)]}$. [mex126]
3. The only central force potentials for which the apsidal angle of nearly circular orbits is independent of the radius are the power-law potentials $V(r) = -\kappa/r^\alpha, \alpha < 2, \alpha \neq 0$ and the logarithmic potential $V(r) = \kappa \ln r$. The value of the apsidal angle is $\Delta\vartheta = \pi/\sqrt{2-\alpha}$, where the value $\alpha = 0$ pertains to the logarithmic potential. [mex127]
4. For central force potentials with $\lim_{r\to\infty} V(r) = \infty$, the apsidal angle has the property $\lim_{E\to\infty} \Delta\vartheta = \pi/2$. [mex128] [mex129]
5. For power-law central force potentials $V(r) = -\kappa/r^\alpha, 0 \leq \alpha < 2$, the apsidal angle has the property $\lim_{E\to-\infty} \Delta\vartheta = \pi/(2-\alpha)$. [mex130]

Proof of Bertrand's theorem:

- Closed orbits require $\Delta\vartheta = 2\pi(m/n)$ for integer m, n.
- Lemma 3 restricts the class of potentials with no open bounded orbits to potentials (a) $V(r) = \kappa' r^{-\alpha}, \alpha < 0$, (b) $V(r) = -\kappa/r^\alpha, 0 < \alpha < 2$, (c) $V(r) = \kappa \ln r$ (representing $\alpha = 0$).
- For the cases $\alpha < 0$, lemma 4 requires $\pi/\sqrt{2-\alpha} = \pi/2$, which rules out all exponents except $\alpha = -2$ (harmonic oscillator). The apsidal angle is $\Delta\vartheta = \pi/2$ for all orbits of this system.
- For the cases $0 \leq \alpha < 2$, lemma 5 requires $\pi/\sqrt{2-\alpha} = \pi/(2-\alpha)$, which rules out all exponents except $\alpha = 1$ (Kepler system). The apsidal angle is $\Delta\vartheta = \pi$ for all orbits of this system.