Orbital Differential Equation [mln46]

Equation of motion for radial motion: $m\ddot{r} - \dfrac{\ell^2}{mr^3} = F(r), \quad F(r) = -V'(r).$

Angular velocity: $\dot{\vartheta} = \dfrac{\ell}{mr^2}.$

Use new radial variable: $u \equiv \dfrac{1}{r}.$

$$\Rightarrow \frac{dr}{dt} = \frac{d}{dt}\frac{1}{u} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\vartheta}\frac{d\vartheta}{dt} = -\frac{\ell}{m}\frac{du}{d\vartheta}.$$

$$\Rightarrow \frac{d^2r}{dt^2} = -\frac{\ell}{m}\frac{d}{dt}\left(\frac{du}{d\vartheta}\right) = -\frac{\ell}{m}\frac{d^2u}{d\vartheta^2}\frac{d\vartheta}{dt} = -\left(\frac{\ell}{m}\right)^2 u^2\frac{d^2u}{d\vartheta^2}.$$

$$\Rightarrow \quad -\frac{\ell^2}{m}u^2\frac{d^2u}{d\vartheta^2} - \frac{\ell^2}{m}u^3 = F\left(\frac{1}{u}\right).$$

Orbital differential equation: $\dfrac{d^2u}{d\vartheta^2} + u = -\dfrac{m}{\ell^2u^2}F\left(\dfrac{1}{u}\right).$

Initial conditions: $u(0) = 1/r_{min},\ 1/r_{max}, \quad u'(0) = 0.$

Like the orbital integral, the orbital differential equation describes the relation between the radial and angular coordinates of an orbit, a relation from which the variable 'time' has been eliminated.

While the orbital integral is most useful for calculating orbits of a given central force potential, the orbital differential equation is particularly useful for finding central force potentials in which given orbits are realized.

Applications:

- Kepler problem [mex48]
- Exponential spiral orbit [mex49]
- Circular orbit through center of force [mex50]
- Linear spiral orbit [mex52]