Heavy symmetric top: general solution [mln47]

Lagrangian: $L = T(\theta, \dot\phi, \dot\theta, \dot\psi) - V(\theta)$. The coordinates ϕ, ψ are cyclic.

$$T = \frac{1}{2}I_\perp(\omega_1^2+\omega_2^2)+\frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_\perp(\sin^2\theta\dot\phi^2+\dot\theta^2)+\frac{1}{2}I_3(\cos\theta\dot\phi+\dot\psi)^2, \quad V = mg\ell\cos\theta.$$

Conserved generalized momenta:

$$\alpha_\phi \equiv \frac{\partial L}{\partial\dot\phi} = (I_\perp \sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\cos\theta\dot\psi = \text{const.}$$

$$\alpha_\psi \equiv \frac{\partial L}{\partial\dot\psi} = I_3(\dot\psi + \cos\theta\dot\phi) = I_3\omega_3 = \text{const} \;\Rightarrow\; \omega_3 = \text{const.}$$

$$\Rightarrow\; \dot\phi = \frac{\alpha_\phi - \alpha_\psi\cos\theta}{I_\perp\sin^2\theta}, \quad \dot\psi = \frac{\alpha_\psi}{I_3} - \frac{(\alpha_\phi - \alpha_\psi\cos\theta)\cos\theta}{I_\perp\sin^2\theta}.$$

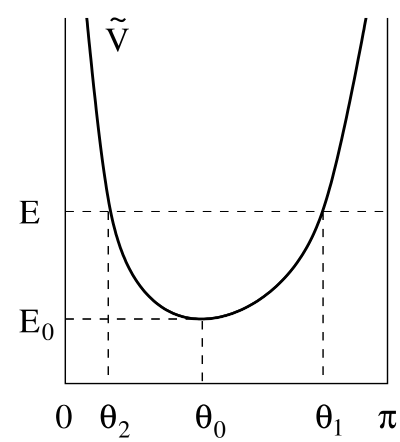
Routhian function: $R(\theta, \dot\theta; \alpha_\phi, \alpha_\psi) = \tilde T(\dot\theta) - \tilde V(\theta)$.

$$\tilde T(\dot\theta) = \frac{1}{2}I_\perp\dot\theta^2, \quad \tilde V(\theta) = \frac{\alpha_\psi^2}{2I_3} + \frac{(\alpha_\phi - \alpha_\psi\cos\theta)^2}{2I_\perp\sin^2\theta} + mg\ell\cos\theta.$$

Conserved energy: $E = \tilde T(\dot\theta) + \tilde V(\theta) = \text{const.}$

Solution by quadrature: $\displaystyle \frac{d\theta}{dt} = \sqrt{\frac{2}{I_\perp}\left[E - \tilde V(\theta)\right]}$.

- Nutation: $\displaystyle t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_\perp}\left[E - \tilde V(\theta)\right]}}$.
- Precession: $\displaystyle \phi(t) = \int dt\, \dot\phi(t)$.
- Rotation: $\displaystyle \psi(t) = \int dt\, \dot\psi(t)$.



Specification of general solution:

- integrals of the motion $\alpha_\psi,\ \alpha_\phi,\ E$,
- starting values $\theta_s,\ \phi_s,\ \psi_s$.

Physical solution for given α_ψ, α_ϕ requires $E \geq E_0 = \tilde V(\theta_0)$.

For energies $E > E_0$ the angle of inclination θ oscillates between θ_1 and θ_2.