Newtonian mechanics in the presence of holonomic constraints

Equations of motion: \( m_i \ddot{r}_i = F_i + Z_i, \quad i = 1, \ldots, N. \)
- \( F_i \): applied forces (known).
- \( Z_i \): forces of constraint (unknown).

Equations of constraint: \( f_j(r_1, \ldots, r_N) = 0, \quad j = 1, \ldots, k \) (scleronomic).

The solution of this dynamical problem within the framework of Newtonian mechanics proceeds as follows:

- The number of degrees of freedom is reduced to \( 3N - k \).
- The number of equations of motion is \( 3N \) with \( 6N \) unknowns \((x_i, y_i, z_i, Z_{ix}, Z_{iy}, Z_{iz}), \quad i = 1, \ldots, N\).
- The geometrical restrictions imposed by the constraints on the orbit yield \( 3N \) additional relations between the unknowns. Among them are the \( k \) equations of constraint.
- A unique solution depends on \( 6N \) initial conditions. Because of the constraints, the number of independent initial conditions is smaller than \( 6N \).
- The reduction of the number of degrees of freedom from \( 3N \) to \( 3N - k \) can be taken into account by introducing \( n = 3N - k \) generalized coordinates \( q_1, \ldots, q_n \) such that the functions \( r_i(q_1, \ldots, q_n), \quad i = 1, \ldots, N \) satisfy the equations of constraint.
- The number of unknowns is thus reduced to \( 6N - k \). The number of equations of motion stays at \( 3N \). The number of additional relations due to the constraints is reduced to \( 3N - k \).

Examples:

- Plane pendulum I [mex132]
- Heavy particle sliding inside cone I [mex133]