Relativity of Space and Time

Frame $S'$ moves with velocity $v$ relative to frame $S$.
Clock is at rest in frame $S'$.
Distance traveled by signal in frame $S'$: $2\ell_0$ (proper length $\ell_0$).
Time period measured in frame $S'$: $\Delta\tau = 2\ell_0/c$ (proper time).

**Time dilation**

Distance traveled by signal in $S$: $2\sqrt{\ell_0^2 + (v\Delta t/2)^2} = c\Delta t$.

Time period measured in $S$: $\Delta t = \frac{2\ell_0/c}{\sqrt{1-v^2/c^2}} = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}}$.

**Length contraction**

Distance traveled by signal in $S$: $c(\Delta t_1 + \Delta t_2) = (\ell + v\Delta t_1) + (\ell - v\Delta t_2)$.

Time period measured in $S$: $\Delta t = \Delta t_1 + \Delta t_2 = \frac{\ell}{c-v} + \frac{\ell}{c+v} = \frac{2\ell_0/c}{1-v^2/c^2}$.

Comparison with proper time and length: $\Delta t = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}} = \frac{2\ell_0/c}{\sqrt{1-v^2/c^2}}$.

Length contraction: $\ell = \ell_0\sqrt{1-v^2/c^2}$.