Length Contraction Paradox [mln53]

Consider two frames S and S' in relative motion with velocity v. Jack is at rest in S and Jill is at rest in S'.

Jack is holding a rod of proper length ℓ_0 in his hand. When Jill measures the length of Jack's rod on the move, how do we reconcile her result with that of Jack's own measurement?

As the two pass by each other, Jill records her measurement by holding her hands a (contracted) distance $\ell' = \ell_0\sqrt{1 - v^2/c^2}$ apart. Jack, in turn, measures the distance between Jill's hands, $\ell'' = \ell'\sqrt{1 - v^2/c^2}$, which is further contracted and thus falls short of ℓ_0.

What is missing?
Relativity of simultaneity must be taken into account!

Let Jill mark her measurement by two synchronized clocks coinciding with the endpoints of Jack's rod at time t'_m. From Jill's perspective, the distance between the clocks is $\ell' = \ell_0\sqrt{1 - v^2/c^2}$.

From Jack's perspective, the distance between the clocks is $\ell'' = \ell_0(1-v^2/c^2)$ and the clocks are out of sync by $\Delta t' = \ell' v/c^2$ in S'. This time lag is dilated to $\Delta t = \Delta t'/\sqrt{1 - v^2/c^2} = \ell_0 v/c^2$ in S.

To determine the length of his own rod via Jill's measurement, Jack must mark the positions of Jill's clocks at instances when they record the same time as seen from S. The result, $\ell'' + v\Delta t = \ell_0$, is in agreement with the proper length of his rod.