Linearly Damped Harmonic Oscillator [mln6]

Equation of motion: $m\ddot{x} = -kx - \gamma\dot{x} \;\Rightarrow\; \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0.$

Damping parameter: $\beta \equiv \gamma/2m$; characteristic frequency: $\omega_0 = \sqrt{k/m}.$

Ansatz: $x(t) = e^{rt} \;\Rightarrow\; (r^2 + 2\beta r + \omega_0^2)e^{rt} = 0 \;\Rightarrow\; r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}.$

Overdamped motion: $\Omega_1 \equiv \sqrt{\beta^2 - \omega_0^2} > 0$

Linearly independent solutions: $e^{r_+ t},\; e^{r_- t}.$

General solution: $x(t) = \left(A_+ e^{\Omega_1 t} + A_- e^{-\Omega_1 t}\right) e^{-\beta t}.$

Initial conditions: $A_+ = (\dot{x}_0 - r_- x_0)/2\Omega_1, \quad A_- = (r_+ x_0 - \dot{x}_0)/2\Omega_1.$

Critically damped motion: $\sqrt{\omega_0^2 - \beta^2} = 0, \quad r = -\beta$

Linearly independent solutions: $e^{rt},\; te^{rt}.$

General solution: $x(t) = (A_0 + A_1 t)e^{-\beta t}.$

Initial conditions: $A_0 = x_0, \quad A_1 = \dot{x}_0 + \beta x_0.$

Underdamped motion: $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2} > 0$

Linearly independent solutions: $e^{r_+ t},\; e^{r_- t}.$

General solution: $x(t) = (A\cos\omega_1 t + B\sin\omega_1 t)\, e^{-\beta t} = D\cos(\omega_1 t - \delta)e^{-\beta t}.$

$$D = \sqrt{A^2 + B^2}, \quad \delta = \arctan(B/A).$$

Initial conditions: $A = x_0, \quad B = (\dot{x}_0 + \beta x_0)/\omega_1.$

The dissipative force, $-\gamma\dot{x}$, effectively represents a coupling of one low-frequency oscillator to many high-frequency oscillators.