Linearly Damped Harmonic Oscillator

Equation of motion: \( m\ddot{x} = -kx - \gamma \dot{x} \Rightarrow \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0. \)

Damping parameter: \( \beta \equiv \gamma / 2m; \) characteristic frequency: \( \omega_0 = \sqrt{k/m}. \)

Ansatz: \( x(t) = e^{rt} \Rightarrow (r^2 + 2\beta r + \omega_0^2)e^{rt} = 0 \Rightarrow r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \)

**Overdamped motion:** \( \Omega_1 \equiv \sqrt{\beta^2 - \omega_0^2} > 0 \)

Linearly independent solutions: \( e^{r_+ t}, \ e^{r_- t}. \)

General solution: \( x(t) = (A_+ e^{\Omega_1 t} + A_- e^{-\Omega_1 t}) e^{-\beta t}. \)

Initial conditions: \( A_+ = (\dot{x}_0 - r_+ x_0)/2\Omega_1, \ A_- = (r_+ x_0 - \dot{x}_0)/2\Omega_1. \)

**Critically damped motion:** \( \sqrt{\omega_0^2 - \beta^2} = 0, \ r = -\beta \)

Linearly independent solutions: \( e^{rt}, \ te^{rt}. \)

General solution: \( x(t) = (A_0 + A_1 t)e^{-\beta t}. \)

Initial conditions: \( A_0 = x_0, \ A_1 = \dot{x}_0 + \beta x_0. \)

**Underdamped motion:** \( \omega_1 \equiv \sqrt{\omega_0^2 - \beta^2} > 0 \)

Linearly independent solutions: \( e^{r_+ t}, \ e^{r_- t}. \)

General solution: \( x(t) = (A \cos \omega_1 t + B \sin \omega_1 t) e^{-\beta t} = D \cos(\omega_1 t - \delta) e^{-\beta t}. \)

\[ D = \sqrt{A^2 + B^2}, \quad \delta = \arctan(B/A). \]

Initial conditions: \( A = x_0, \ B = (\dot{x}_0 + \beta x_0)/\omega_1. \)

The dissipative force, \( -\gamma \dot{x}, \) effectively represents a coupling of one low-frequency oscillator to many high-frequency oscillators.