Central Force Motion: Two-Body Problem [mln66]

Mechanical system with six degrees of freedom:

Consider two masses m_1, m_2 interacting via a central force.

Central-force potential: $V(\mathbf{r}_1, \mathbf{r}_2) \equiv V(|\mathbf{r}_1 - \mathbf{r}_2|)$.

Lagrangian of two-body problem: $L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|)$.

Conservation laws inferred from translational and rotational symmetries:

- Energy: $E = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 + V(|\mathbf{r}_1 - \mathbf{r}_2|)$.
- Linear momentum: $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2$.
- Angular momentum: $\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2$.

Reduction to three degrees of freedom:

Center-of-mass position vector: $\mathbf{R} \doteq \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$.

Distance vector: $\mathbf{r} \doteq \mathbf{r}_2 - \mathbf{r}_1$.

Total mass: $M \doteq m_1 + m_2$.

Reduced mass: $m \doteq \frac{m_1 m_2}{m_1 + m_2}$.

Lagrangian (after point transformation):

$$L = L_M(\dot{\mathbf{R}}) + L_m(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}m\dot{\mathbf{r}}^2 - V(|\mathbf{r}|).$$

Center-of-mass motion: $L_M(\dot{\mathbf{R}}) = \frac{1}{2}M\dot{\mathbf{R}}^2$.

- R_x, R_y, R_z are cyclic coordinates.
- Conserved center-of-mass momentum: $\mathbf{P} = M\dot{\mathbf{R}} = \text{const.}$
- Uniform rectilinear center-of-mass motion: $\mathbf{R}(t) = \mathbf{R}_0 + \frac{\mathbf{P}}{M}\,t$.

Effective one-body problem: $L_m(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(|\mathbf{r}|)$.

- Three degrees of freedom.
- Particle of mass m moving in a stationary central potential $V(|\mathbf{r}|)$.