Virial Theorem [mln68]

Consider a system of interacting particles in bounded motion.

Newton's equations of motion: $\dot{\mathbf{p}}_i = m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \quad i = 1, \ldots, N.$

$\mathbf{F}_i$: sum of external and interaction forces acting on particle i.

Definition: $G(t) \doteq \sum_i \mathbf{p}_i \cdot \mathbf{r}_i.$

For bounded motion $G(t)$ is finite.

Time derivative: $\displaystyle \frac{dG}{dt} = \sum_i (\mathbf{p}_i \cdot \dot{\mathbf{r}}_i + \dot{\mathbf{p}}_i \cdot \mathbf{r}_i) = \sum_i m_i |\dot{\mathbf{r}}_i|^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i.$

Kinetic energy: $\displaystyle T = \sum_i \frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2.$

Time average: $\displaystyle \overline{\frac{dG}{dt}} = \frac{1}{\tau}\int_0^\tau dt\, \frac{dG}{dt} = \frac{1}{\tau}\left[G(\tau) - G(0)\right] \stackrel{\tau\to\infty}{\longrightarrow} 0.$

$$\Rightarrow\ 2\overline{T} + \sum_i \overline{\mathbf{F}_i \cdot \mathbf{r}_i} = 0.$$

Virial: $\displaystyle \overline{T} = -\frac{1}{2}\sum_i \overline{\mathbf{F}_i \cdot \mathbf{r}_i}.$

Application to particle in bounded orbit of central-force motion.

Power-law central force potential: $\displaystyle V(r) = -\frac{\kappa}{r^\alpha}.$

$$\overline{T} = -\frac{1}{2}\left(\overline{-r\,\frac{dV}{dr}}\right) = -\frac{1}{2}\alpha\,\overline{V}.$$

- Gravity ($\alpha = 1$): $\overline{T} = -\frac{1}{2}\overline{V}.$
- Harmonic oscillator ($\alpha = -2$): $\overline{T} = \overline{V}.$