Limit Cycles [mln74]

Not all attractors in 2D phase flow are fixed points.
There exists exactly one other type of attractor: the limit cycle.

Example: Flow in (x, y)-plane with circular limit cycle.

Equations of motion in polar coordinates (r, θ):

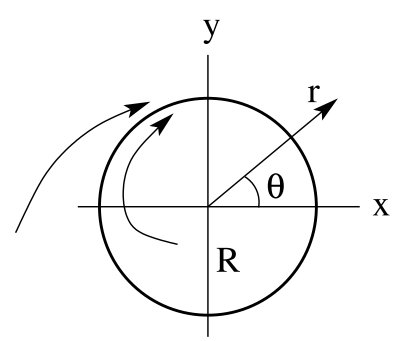
$$\dot{r} = -\alpha r(r - R), \qquad \dot{\theta} = \omega = \text{const.}$$

Periodic trajectory: $r(t) = R = \text{const.}, \quad \theta(t) = \theta_0 + \omega t.$

Linearized radial equation of motion for $|r(t) - R| \doteq s \ll R$:

$$\dot{s} = -\alpha R s \quad \Rightarrow \; s(t) = s_0 e^{-\alpha R t}.$$

$\Rightarrow$ Periodic trajectory is an attractor (limit cycle).



Note presence of fixed point at $r = 0$:

Cartesian coordinates: $x = r\cos\theta, \quad y = r\sin\theta.$

Linear anlaysis of equations of motion predict spiral repellor:

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta = -\alpha\left(\sqrt{x^2+y^2} - R\right)x - \omega y,$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -\alpha\left(\sqrt{x^2+y^2} - R\right)y + \omega x,$$

$$\Rightarrow \; \mathbf{A} = \begin{pmatrix} \alpha R & -\omega \\ +\omega & \alpha R \end{pmatrix} \quad \Rightarrow \; \lambda = \alpha R \pm i\omega.$$