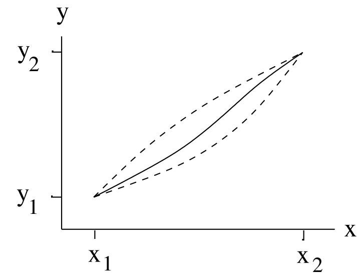
Calculus of Variation [mln78]

Given a functional $f(y, y'; x)$ with $y' = dy/dx$, determine the path $y(x)$ between fixed endpoints $y(x_1) = y_1$, $y(x_2) = y_2$ such that the following integral is an extremum:

$$J = \int_{x_1}^{x_2} dx\, f\big(y(x), y'(x); x\big).$$



Variation of path: $y(x, \alpha) = y(x, 0) + \alpha\eta(x)$ with $\eta(x_1) = \eta(x_2) = 0$.

Parametrized integral: $J(\alpha) = \int_{x_1}^{x_2} dx\, f\big(y(x,\alpha), y'(x,\alpha); x\big)$.

Extremum condition: $\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0$ for arbitrary $\eta(x)$.

Differentiation: $\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)$.

Integration by parts (second term):

$$\int_{x_1}^{x_2} dx\, \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x \partial \alpha} = \left[\frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}.$$

Substitute and use $(\partial y/\partial\alpha)_{\alpha=0} = \eta(x)$:

$$\Rightarrow \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x) = 0.$$

Requirement that integral must vanish for arbitrary $\eta(x)$ implies

Euler's equation: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$.

Notation used in calculus of variation:

Variation of path: $\left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} d\alpha \doteq \delta y$.

Variation of integral: $\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} d\alpha \doteq \delta J$.

$$\Rightarrow \delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y = 0.$$