Bounded Orbits Open or Closed [mln79]

Consider an effective potential $\tilde{V}(r) = V(r) + \ell^2/(2mr^2)$ for the radial part of a central force motion as shown.

The radial coordinate r oscillates between r_P (periapsis) and r_A (apsis).

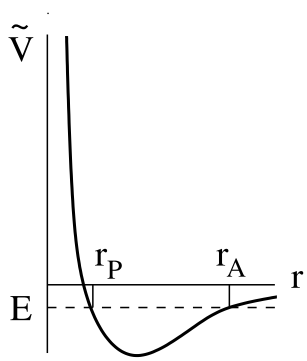
Between successive instances of $r = r_P$ and $r = r_A$ the angular coordinate ϑ always advances the same amount $\Delta\vartheta$.

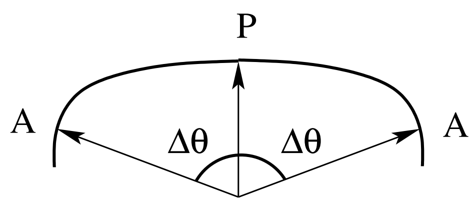
Apsidal vectors: position vectors $\mathbf{r}$ with $|\mathbf{r}| = r_P$ or $|\mathbf{r}| = r_A$.

Orbits are reflection symmetric at apsidal vectors. Hence the complete orbit can be constructed from one segment between successive apsidal vectors.

Apsidal angle: $\Delta\vartheta = \displaystyle\int_{r_P}^{r_A} dr \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}}.$

Condition for closed orbit: $\Delta\vartheta/2\pi$ must be a rational number.





Examples of closed bounded orbits:

- $V(r) = -\dfrac{\kappa}{r} \quad \Rightarrow \quad \vartheta - \vartheta_0 = \arccos \dfrac{\frac{\ell^2}{m\kappa r} - 1}{\sqrt{1 + \frac{2E\ell^2}{m\kappa^2}}} \quad \Rightarrow \quad \Delta\vartheta = \pi.$
- $V(r) = \dfrac{1}{2}kr^2 \quad \Rightarrow \quad \vartheta - \vartheta_0 = \dfrac{1}{2}\arccos \dfrac{\frac{\ell}{mr^2} - \frac{E}{\ell}}{\sqrt{\frac{E^2}{\ell^2} - \frac{k}{m}}} \quad \Rightarrow \quad \Delta\vartheta = \dfrac{\pi}{2}.$

Bertrand's theorem [mln44] proves that only for these two potentials are all bounded orbits closed.