Bounded Orbits Open or Closed

Consider an effective potential \( \tilde{V}(r) = V(r) + \ell^2/(2mr^2) \) for the radial part of a central force motion as shown.

The radial coordinate \( r \) oscillates between \( r_P \) (periapsis) and \( r_A \) (apsis).

Between successive instances of \( r = r_P \) and \( r = r_A \) the angular coordinate \( \vartheta \) always advances the same amount \( \Delta \vartheta \).

Apsidal vectors: position vectors \( \mathbf{r} \) with \( |\mathbf{r}| = r_P \) or \( |\mathbf{r}| = r_A \).

Orbits are reflection symmetric at apsidal vectors. Hence the complete orbit can be constructed from one segment between successive apsidal vectors.

Apsidal angle: \( \Delta \vartheta = \int_{r_P}^{r_A} dr \frac{\ell/mr^2}{\sqrt{\frac{2}{m} \left[ E - V(r) - \frac{\ell^2}{2mr^2} \right]}} \).

Condition for closed orbit: \( \Delta \vartheta / 2\pi \) must be a rational number.

Examples of closed bounded orbits:

1. \( V(r) = -\frac{\kappa}{r} \Rightarrow \vartheta - \vartheta_0 = \arccos \frac{\ell^2}{mkr} - 1 \sqrt{1 + \frac{2E\ell^2}{mk^2}} \Rightarrow \Delta \vartheta = \pi. \)

2. \( V(r) = \frac{1}{2}kr^2 \Rightarrow \vartheta - \vartheta_0 = \frac{1}{2} \arccos \frac{\ell}{\sqrt{E^2/k^2} - \frac{k}{m}} \Rightarrow \Delta \vartheta = \frac{\pi}{2}. \)

Bertrand’s theorem \([\text{mln}44]\) proves that only for these two potentials are all bounded orbits closed.