Principal Axes of Inertia [mln80]

The inertia tensor **I** is real and symmetric. Hence it has real eigenvalues.

The orthogonal transformation which diagonalizes the inertia tensor is a rotation of the body coordinate axes to the directions of the *principal axes*.

The eigenvalue problem,

$$\mathbf{I} \cdot \vec{\omega}_k = I_k \, \vec{\omega}_k, \quad k = 1, 2, 3,$$

amounts to a system of linear, homogeneous equations,

$$\begin{aligned}
I_{11}\omega_{1k} + I_{12}\omega_{2k} + I_{13}\omega_{3k} &= I_1\omega_{1k} \\
I_{21}\omega_{1k} + I_{22}\omega_{2k} + I_{23}\omega_{3k} &= I_2\omega_{2k} \\
I_{31}\omega_{3k} + I_{32}\omega_{3k} + I_{33}\omega_{3k} &= I_3\omega_{3k},
\end{aligned}$$

where the principal moments of inertia $I_k, \; k = 1, 2, 3$ are the roots of the characteristic polynomial,

$$\begin{vmatrix} I_{11} - I_k & I_{12} & I_{13} \\ I_{21} & I_{22} - I_k & I_{23} \\ I_{31} & I_{32} & I_{33} - I_k \end{vmatrix} = 0.$$

The three eigenvectors $\vec{\omega}_k, \; k = 1, 2, 3$ have undetermined magnitude.
If normalized, we construct an orthonormal matrix from them:

$$\Omega = (\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3) = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}.$$

In matrix notation the transformation to principal axes looks as follows:

$$\mathbf{I} \cdot \Omega = \Omega \cdot \bar{\mathbf{I}} \qquad \Rightarrow \; \Omega^T \cdot \mathbf{I} \cdot \Omega = \Omega^T \cdot \Omega \cdot \bar{\mathbf{I}} = \bar{\mathbf{I}},$$

where

$$\mathbf{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}, \qquad \bar{\mathbf{I}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix},$$

$$\Omega^T \cdot \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$