Heavy symmetric top: steady precession [mln81]

Special case: $E = E_0 \;\Rightarrow\; \theta = \theta_0 = \text{const} \;\Rightarrow\; \dot\phi = \text{const},\; \dot\psi = \text{const}.$

Steady angle of inclination θ_0 determined by condition $(d\tilde{V}/d\theta)_{\theta_0} = 0$.
$\Rightarrow$ Quadratic equation for $\beta_0 \doteq \alpha_\phi - \alpha_\psi \cos\theta_0$:

$$(\cos\theta_0)\beta_0^2 - (\alpha_\psi \sin^2\theta_0)\beta_0 + mg\ell I_\perp \sin^4\theta_0 = 0.$$

Solution: $\beta_0^\pm = \dfrac{\alpha_\psi \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \dfrac{4mg\ell I_\perp \cos\theta_0}{\alpha_\psi^2}}\,\right].$

Interpretation: For given θ_0 and α_ψ there exist two values $\alpha_\phi^\pm$ for which steady precession is realized.

Distinguish frequencies of fast precession (+) and slow precession (−):

$$\dot\phi_0^\pm = \frac{\beta_0^\pm}{I_\perp \sin^2\theta_0}.$$

Distinguish hanging top ($\theta_0 > \pi/2$) and standing top ($\theta_0 < \pi/2$):

- $\theta_0 > \pi/2$: Steady precession exists without restrictions on α_ψ.
- $\theta_0 < \pi/2$: Steady precession requires that angular velocity about figure axis exceeds threshold value:

$$\alpha_\psi^2 \geq 4mg\ell I_\perp \cos\theta_0 \quad \Rightarrow\; \omega_3 = \frac{\alpha_\psi}{I_3} \geq \frac{2}{I_3}\sqrt{mg\ell I_\perp \cos\theta_0}.$$

Consider fast top ($\alpha_\psi \gg 2\sqrt{mg\ell I_\perp}$):

$$\beta_0^\pm \simeq \frac{\alpha_\psi \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm 1 \mp \frac{2mg\ell I_\perp \cos\theta_0}{\alpha_\psi^2}\right].$$

- Fast precession: $\beta_0^\pm \simeq \dfrac{\alpha_\psi \sin^2\theta_0}{\cos\theta_0} \quad \Rightarrow\; \dot\phi_0^+ \simeq \dfrac{I_3\omega_3}{I_\perp \cos\theta_0}.$
- Slow precession: $\beta_0^\pm \simeq \dfrac{mg\ell I_\perp \sin^2\theta_0}{\alpha_\psi} \quad \Rightarrow\; \dot\phi_0^- \simeq \dfrac{mg\ell}{I_3\omega_3}.$