Variational Principle in Phase Space [mln83]

Hamilton's principle: variations in configuration space.

$$\delta J \doteq \delta \int_{t_1}^{t_2} dt\, L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = 0,$$

where $\delta q_i = 0$ at t_1 and t_2.

$\Rightarrow$ Lagrange equations: $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, \ldots, n.$

Derivation: [mln78], [msl20].

Modified Hamilton's principle: variations in phase space.

$$\delta J \doteq \delta \int_{t_1}^{t_2} dt\, \left[\sum_{i=1}^{n} p_i \dot{q}_i - H(q_1, \ldots, q_n; p_1, \ldots, p_n; t)\right] = 0,$$

where $\delta q_i = 0$ and $\delta p_i = 0$ at t_1 and t_2.

$\Rightarrow$ Canonical equations: $\dot{q}_i = \dfrac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n.$

Derivation:

$$\delta J = \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[p_i \delta \dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial q_i}\,\delta q_i - \frac{\partial H}{\partial p_i}\,\delta p_i\right] = 0;$$

use $\displaystyle\int_{t_1}^{t_2} dt \sum_{i=1}^{n} p_i \delta \dot{q}_i = \underbrace{\left[\sum_{i=1}^{n} p_i \delta q_i\right]_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \dot{p}_i \delta q_i;$

$$\Rightarrow \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right] = 0.$$