Charged Particle in Electromagnetic Field [mln86]

Lorentz force: $\mathbf{F} = e\,\mathbf{E} + \frac{e}{c}\,\mathbf{v}\times\mathbf{B}$.

Electric field: $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$.

Magnetic field: $\mathbf{B} = \nabla\times\mathbf{A}$.

Velocity-dependent potential energy: $U(\mathbf{r},\mathbf{v},t) = e\phi(\mathbf{r},t) - \frac{e}{c}\,\mathbf{v}\cdot\mathbf{A}(\mathbf{r},t)$.

Lagrangian: $L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m|\mathbf{v}|^2 - U(\mathbf{r},\mathbf{v},t)$.

Lagrange equations for $\mathbf{r} = (x_1, x_2, x_3)$, $\mathbf{v} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$:

$$m\ddot{x}_1 + \frac{e}{c}\frac{dA_1}{dt} = e\left(-\frac{\partial\phi}{\partial x_1} + \frac{\dot{x}_1}{c}\frac{\partial A_1}{\partial x_1} + \frac{\dot{x}_2}{c}\frac{\partial A_2}{\partial x_1} + \frac{\dot{x}_3}{c}\frac{\partial A_3}{\partial x_1}\right) \qquad \text{etc.}$$

Use $\frac{dA_1}{dt} = \frac{\partial A_1}{\partial t} + \mathbf{v}\cdot\nabla A_1 = \frac{\partial A_1}{\partial t} + \dot{x}_1\,\frac{\partial A_1}{\partial x_1} + \dot{x}_2\,\frac{\partial A_1}{\partial x_2} + \dot{x}_3\,\frac{\partial A_1}{\partial x_3}$.

$$\begin{aligned} \Rightarrow\ m\ddot{x}_1 &= e\left[-\frac{\partial\phi}{\partial x_1} - \frac{1}{c}\frac{\partial A_1}{\partial t} + \frac{\dot{x}_2}{c}\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) + \frac{\dot{x}_3}{c}\left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right)\right] \\ &= eE_x + \frac{e}{c}\,(\mathbf{v}\times\mathbf{B})_x \quad \text{etc.} \end{aligned}$$

Generalized momenta: $p_i = \frac{\partial L}{\partial\dot{x}_i} = m\dot{x}_i + \frac{e}{c}\,A_i$.

- p_i: canonical momenta.
- $m\dot{x}_i$: kinetic momenta

Hamiltonian: $H(\mathbf{r},\mathbf{p},t) = \sum_{i=1}^{3} p_i\dot{x}_i - L = \frac{1}{2m}\left|\mathbf{p} - \frac{e}{c}\,\mathbf{A}(\mathbf{r},t)\right|^2 + e\phi(\mathbf{r},t)$.

Relativistic mechanics:

- Momenta: [mln63]
- Hamiltonian: $H(\mathbf{r},\mathbf{p},t) = \sqrt{\left|\mathbf{p} - \frac{e}{c}\,\mathbf{A}(\mathbf{r},t)\right|^2 + m_0^2c^4} + e\phi(\mathbf{r},t)$.