Properties of the Hamiltonian [mln87]

How is the Hamiltonian constructed from kinetic energy and potential energy? When does it represents the total energy? When is it conserved?

Here is a list of some answers:

- If the Hamiltonian does not depend explicitly on time then it is a conserved quantity: $H(q_1, \ldots, q_n; p_1, \ldots, p_n) = \text{const.}$

$$\frac{dH}{dt} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j}\,\dot{q}_j + \frac{\partial H}{\partial p_j}\,\dot{p}_j \right) = \sum_{j=1}^{n} (-\dot{p}_j \dot{q}_j + \dot{p}_j \dot{q}_j) = 0.$$

- If $T(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ is the kinetic energy and $V(q_1, \ldots, q_n; t)$ the potential energy of a Lagrangian $L = T - V$, then the Hamiltonian is equal to the total energy:

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = T + V = E(t), \quad \text{where } p_j \doteq \frac{\partial L}{\partial \dot{q}_j}.$$

- Suppose that some of the generalized coordinates $q_1, \ldots, q_n$ are subject to holonomic constraints. Then $H = T + V$ only holds if all those constraints are scleronomic, i.e. time-independent [mex81].
- Depending on the nature of the dynamical system and the choice of coordinates, the Hamiltonian may represent the total energy or a conserved quantity or both or neither [mex77].
- The property $H \neq T + V$ occurs in the presence of velocity-dependent potentials [mln85]. The motion of a charged particle in a static magnetic field is a prominent example [mln86].
- In the presence of time-dependent fields, the conceptual framework used here quickly shows its limitations, because such fields themselves can transport momentum and energy.